Naiveté and sophistication in games
v0.03

Zsombor Z. Méder[a,*], János Flesch[b], Ronald Peeters[a]

[a]*Department of Economics, Maastricht University, The Netherlands*
[b]*Department of Quantitative Economics, Maastricht University, The Netherlands*

Abstract

Keywords: dynamic inconsistency; naiveté; sophistication; stochastic games
JEL:

1. Motivation

This paper studies strategic interactions of dynamically inconsistent players. In order to simplify the discussion, we limit our attention to two-player games. For now, we study only extensive-form games with perfect information, not allowing for simultaneous moves.

2. Related literature

3. Preliminaries

3.1. The game

Our game is then made up of the following:

- the set of players $N = \{1, 2\}$;
- the set of player types $X = \{Rr, Rn, Nr, Nn, S\}$ – for a detailed explanation and discussion of these, see Sections 3.4 and 3.5 below.
- the set of time periods $T = \{0, 1, 2, \dots\}$ (possibly bounded from above);
- the set of states Ω, with $\bar{\omega} \in \Omega$ as the initial state;
- the sets of available pure actions A^i_ω (the pure action set for player i in state ω. As we assume no simultaneous moves, for each ω, there exists exactly one player $i \in N$ such that A^i_ω is nonempty. For simplicity, we require that $A^i_\omega \bigcap A^j_{\omega'} = \emptyset$ whenever $\omega \neq \omega'$. Finally, the union of all action sets is denoted by A;

[*]Corresponding author
Email addresses:
`z.meder@maastrichtuniversity.nl` (Zsombor Z. Méder),
`j.flesch@maastrichtuniversity.nl` (János Flesch),
`r.peeters@maastrichtuniversity.nl` (Ronald Peeters)

- the payoff functions $u^i : A \to \mathbb{R}$. These represent the payoff for player i associated with taking particular actions;
- transition probabilities $m_\omega : A^i_\omega \to \Delta(\Omega)$, with $m_\omega(\omega'|a_\omega)$ denoting the probability to transit from state ω to state ω' when action a_ω is chosen;

Our stochastic game with perfect information is thus:

$$\Gamma = \{N, X, \Omega, (A^i_\omega)_{\omega\in\Omega, i\in N}, (u^i)_{i\in N}, (m_\omega)_{\omega\in\Omega}\}.$$

Another way to treat payoffs would be to define the game tree for the extensive form game (see the next subsection), and then attach payoffs or utilities to the terminal nodes. We chose this representation to show the connections between Markov decision problems and games. For finite or infinitely repeated games, the two approaches are equivalent.

3.2. The game tree

We define a history as a sequence of states and actions, just like in a decision problem.

Definition 1. A *history* h has the form $h = (\omega_0, a_{\omega_0}, \ldots, \omega_{t-1}, a_{\omega_{t-1}}, \omega_t)$, with:

- $\omega_j \in \Omega$, for all $j \in \{0, 1, \ldots, t\}$, and $\omega_0 = \bar{\omega}$;
- for all $j \in \{0, 1, \ldots, t-1\}$ there is some $i \in N$ such that $a_{\omega_j} \in A^i_{\omega_j}$;
- $m_{\omega_j}(\omega_{j+1}|a_{\omega_j}) > 0$, for $j \in \{0, 1, \ldots, t-1\}$.

The length of h or current time at h is denoted by $t = t(h)$, and the function $\omega(h) = \omega_{t(h)}$ indicates the current or end state at history h. We use H to refer to the set of all histories.. If player i's action set is nonempty at history h (i.e., $A^i_{\omega(h)} \neq \emptyset$), then we denote this fact by writing $p(h) = i$, and call him the active player (or: "the agent") at h. The other player is called passive.

3.3. Payoffs and utility

3.4. Types of players

We will now introduce the basic types whose interaction we will study: resolutes, naifs and sophisticates. Note that the size of our type set will grow from these three to five once we introduce beliefs in Section3.5. *Resolute* players can commit to a strategy, i.e., they choose a strategy in the first period, and they stick to it until the end. Resolution is usually interpreted as *strength of will*, such that although a resolute player still faces preference change, he is able to stick to a strategy he chose in a previous period, when his preferences were different. The strength-of-will interpretation implies that a resolute player believes (and thus, knows[1]) that he is resolute.

There are some arguments for the possibility of resolute behavior (McClennen 1991). However, in our model, there are no players that are actually resolute, albeit some players believe they are so (see below). Therefore, resolution plays an indirect role here.

Naiveté has several interpretations in the literature. According to one, *naive* players are completely unaware that their preferences will change. Alternatively, they think that – albeit their preferences will change – they can commit to a strategy. We will here go with the second interpretation, as it is more straightforward to model than

[1] We interpret knowledge as true belief in this paper.

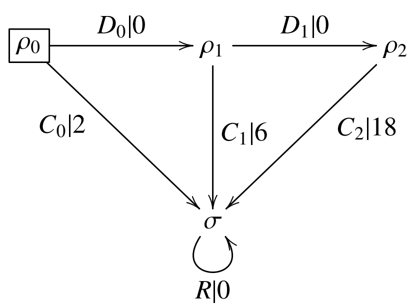
unawareness. That is, naifs are really naive, but they believe they are resolute, that is, they believe a falsehood.[2] This unreasonable belief can lead naifs to choose strategies that they later deviate from, and are thus prone to exhibit not only dynamically inconsistent preferences, but dynamically inconsistent choice, so that their plans for future actions are not adhered to later.

Many papers examine dynamically inconsistent players who are naive. An interesting aspect of naiveté on the current, simple account is that it is not assumed that there is a self-correcting tendency among such players – naifs stay naive for the entire duration of the game. Thus, a naive player sets out with a certain plan for future actions, then chooses some different action according to his new preferences, but doesn't realize this dichotomy. Alternatively, even if he realizes that he did not stick to his old plan, he might believe that he *will* stick to his plan that he is forming now. This is at odds with how equilibria are usually interpreted in a game setting. In an equilibrium, it is assumed that all players have correct beliefs about other players. The reason is that otherwise, a player might observe a behavior that are at odds with their beliefs, so that their beliefs are said to be out-of-equilibrium. However, a naive player might observe an action (chosen by himself) every single period (except the first one) that is at odds with previous beliefs about what action he himself will pick. We see that a naive player is never in equilibrium in the game-theoretic usual sense. A major aim of this paper is to resolve this discrepancy.

Sophisticated players are aware of the problem of their changing preferences and the commitment problem. They first consider the preferences they will have, reason about the choices they are going to make, and then give a best response to that future behavior. Sophisticates thus choose an (intra-personal) subgame-perfect equilibrium, and choose an action corresponding to this equilibrium, – correctly – assuming they will also do so in the future. This means that a sophisticated player knows he is sophisticated, and that he will be so in all periods.

Sophistication is also commonly analyzed in the context of dynamic inconsistency. In this paper, we do not consider the relative plausibility of assuming naiveté or sophistication. All we assume is that naifs beliefs that they are resolute are fundamentally wrong. Moreover, we only deal with naive and sophisticated players, and their interactions, ignoring resolution altogether.

We use the simple decision problem represented on Figure 1 to illustrate how the behavior of the three types differ.



$\beta = 0.5, \delta = 0.5$

Figure 1: A simple decision problem where resolution, naiveté and sophistication lead to different behavior.

The problem concerns the consumption of a single good, whose quality improves over time, but which has to consumed at most after two periods. "Consume" and "delay" choices are represented by C_i, with i indicating the

[2]In fact, the term "naive" can be traced to such players having a false belief about themselves – or, on the unawareness interpretation, on having no access to information about themselves.

period. The initial state, indicated by a box, is ρ_0. We assume that the utility function of each agent is quasi-hyperbolic, with $\beta = \delta = 0.5$. By a simple calculation, we get:

- $U^{h_0}(s^{h_0}_{C_0}) = 2$;
- $U^{h_0}(s^{h_0}_{C_1}) = 0 + \frac{1}{2} \cdot \frac{1}{2} \cdot 6 = 1.5$;
- $U^{h_0}(s^{h_0}_{C_2}) = 0 + 0 + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot 18 = 2.25$;
- $U^{h_1}(s^{h_1}_{C_1}) = 6$;
- $U^{h_1}(s^{h_1}_{C_2}) = 0 + \frac{1}{2} \cdot \frac{1}{2} \cdot 18 = 4.5$.

This means that the agent at h_0 is best off by delaying the consumption for two periods, and the choosing C_2. On the other hand, the agent at h_1 is better off choosing C_1 than delaying consumption, and then choosing C_2.

How do our types behave in this situation? A resolute decision maker at h_0 would be able to choose and commit[3] to the strategy of delaying consumption for two periods. After choosing D_0, and reaching state ρ_1, he would face temptation to consume immediately. However, through his strength of will, he would overcome this temptation, sticking to his initial plans, and delay consumption until the second period.

A naive decision maker would also plan to consume after two periods. Thinking himself resolute, he believes he will be able to do so. Therefore, in the first period, he decides to delay his consumption. However, after that, he yields to the temptation, and takes the consumption choice C_1.

A sophisticated decision maker will start thinking about his future behavior first. Noticing the temptation he will face in state ρ_1, he realizes he will choose C_1 if he ever reaches that state. Thus, in the original state ρ_0, he can realistically only make a choice between C_0 and C_1. Because the first brings him higher utility, he choose C_0, and consumes immediately.

We see how the three different types lead to different behavior in our simple decision problem. While for a single-player decision problem, an understanding of these types suffices to make behavioral predictions and welfare comparisons, in a game setting, the specification of beliefs of players about other players is necessary to

3.5. Modelling beliefs

To be able to speak formally about the beliefs of players, we introduce a modal language that is able to express player types, and player's beliefs about them. In total, we deal with two different types of naif (Nr, Nn); for this reason, we also need two types of resolutes (Rr, Rn). Finally, we will have a single type for sophistication (S).

Technically, since each player is just the collection of its agents at various histories, it would be more precise to speak about agents being of a certain type, and holding certain beliefs, such as: "the agent at h believes that the agent at h' is naive". However, we assume that:

- Beliefs (of any order) about all agents belonging to the same player coincide.
- All agents of the same player have the same beliefs about other players.

[3] Note that this in this paper, we assume that commitment and thus resolute behavior is in fact not possible, it only exists as the self-perception of the naive.

We now want to add axioms characterizing the beliefs of each player. The main problem is that the classical equilibrium notion whereby the beliefs of all players are correct is incompatible with naiveté.[4] A naive player can thus never believe he is naive – he believes he is resolute. The problem is then how to construct a belief hierarchy that is consistent, but reflects this unawareness of the own type.

Assumption 1. A sophisticated player knows he is sophisticated.

Assumption 2. A naive player believes he is resolute.

Assumption 3. Each player always holds *correct* beliefs about the identity of the other player and any higher-order beliefs concerning the other player.[5]

Assumption 4. Sophisticates hold correct beliefs about the other's beliefs (any any higher-order beliefs) concerning themselves.

Assumption 5. Naive players can be can be either right or wrong about the beliefs of other player concerning their own identity.

In important concern for naive players is that if player 1 believes player 2 is naive, then player 1 will also believe player 2 does *not* believe he is naive. On the other hand, what is the second-order belief of a naive player 2? Either he must (correctly) believe that player 1 believes that he is naive (and he would assume player 1 is *wrong*, although player 1 is, in fact, *right*) or he must (incorrectly) believe that player 1 believes him to be resolute (then, he would assume player 1 is *right* - although player 1 actually holds a different belief). On the other hand, sophistication, as a concept of intra-personal Nash-equilibrium, and requires self-awareness. Thus, if player 1 believes player 2 is sophisticated, then player 1 will also believe player 2 is aware of his sophistication.

The types and their beliefs are represented on Figure 2. Where a type is indicated by two letters, the first letter represents the real type of the player, whereas the second letter indicates the second-order belief of the same player. For instance, Nr is a naive type who believes that his opponent believes that he is naive. We note that type *Rn* is purely used for modelling purposes.

Our diagram allows for a derivation of all belief hierarchies. For instance, suppose that the real state of the world is Nn, Nr. Then, the first-order beliefs of player 1 are given by following the blue arrow, which in this case points to *Rn*, Nr. Thus, player 1 believes that player 1 is of type *Rn* (resolute), and also believes that player 2 is Nr (naive). Player 1's second-order beliefs about player 2 in Nn, Nr can be generated by first following the blue, and then the red arrow. Thus, player 1 believes player 2 believes that the state of the world is Nn, Rr. Thus, player 1 believes player 2 believes player 1 is naive, and player 1 also believes that player 2 believes that player 2 is resolute.

A short consideration of our diagram indicates that Assumptions 1-5 specified above are all satisfied within our model.

[4]Essentially, in the modal logic that works behind our model, we can not assume the truthfulness axiom schema **T** ($B_i(A) \supset A$), as a naive player is assumed to believe something false (namely, that he is not naive).

[5]This is a reflection of the following idea: in equilibrium, each player examines the behavior of *the other*, but not necessarily *himself*. In a sense, (naive) players are behaving like the person criticized in Matthew 7:4 "How can you say to your brother, 'Let me take the speck out of your eye,' when all the time there is a plank in your own eye?"

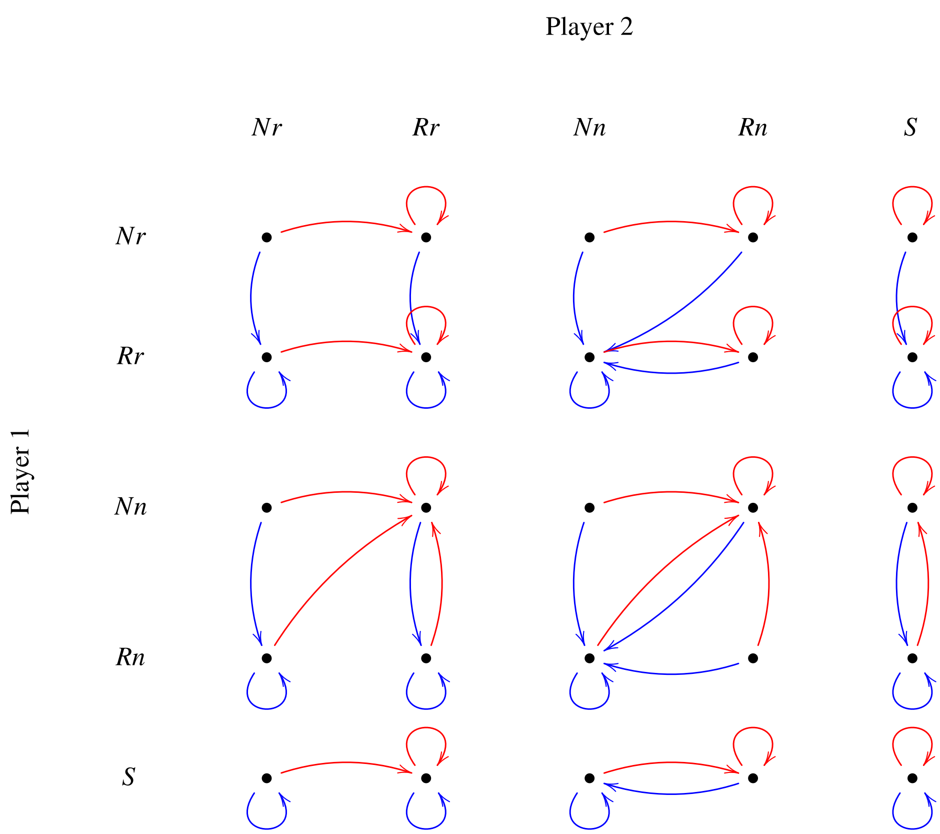


Figure 2: The five types, and their beliefs. Player 1 is indicated in blue, player 2 in red.

4. Equilibria

5. Some examples

6. Summary