Efficient Teamwork

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Abstract

In real-life multi-agent projects, agents often choose actions that are highly inefficient for the project or damaging for other agents because they care only about their own contracts and interests. We show that this can be avoided by the right project management.

We model agents with private workflows including hidden actions and chance events, which can influence each other through publicly observable actions and events. We design an efficient mechanism for this model which is prior-free, incentive-compatible, collusion-resistant, individually rational and avoids free-riders.

Keywords: Mechanism design, Stochastic dynamic mechanisms, Games with incomplete information, Robust mechanism design. JEL: C72, C73.

1 Introduction

Consider the case when we want to hire an agent who cannot guarantee the quality of the result of his task. For example, if there are only two objective quality levels, and the agent reports that he will provide high quality with probability 60%, or low quality with probability 40%, and his fee is \$10. We do not trust his reports about the probabilities. But we know that the difference between our valuations of the high- and low-quality results is \$5, therefore, we should agree the following. If the agent provides a high-quality result, then we pay him \$12, otherwise we pay him \$7. Hereby, on one hand, we are indifferent to whether his probabilities were true, and on the other hand, the agent would get \$10 in expectation if his reported probabilities were true. In this paper, we show a generalization of this well-known technique, when we apply multiple agents who have hidden but interdependent dynamic workflows with in-progress decisions and chance events.

Consider now an example of a project management problem with two players, Alice and Bob. They have two independent tasks to complete, but only the later completion time matters. During the execution of Alice's task, she realizes and reports that she will probably be ready earlier than she initially estimated. We know that it would be very profitable if both Alice and Bob were to finish earlier, therefore, we ask Bob to work faster, even if this causes him extra expense. This would be the efficient decision, but it raises some questions. How much money should we pay them? What if ultimately Alice were to finish only by the original deadline? How can we make the efficient joint behavior incentive-compatible, even though we cannot observe the abilities of the agents and the efforts they made?

In general, we consider multi-agent projects such as building a house. We assume that the agents have separate tasks which may influence each other through verifiable events. For

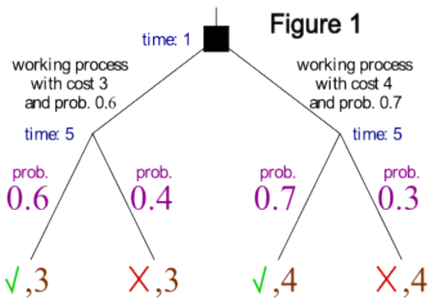


Figure 1

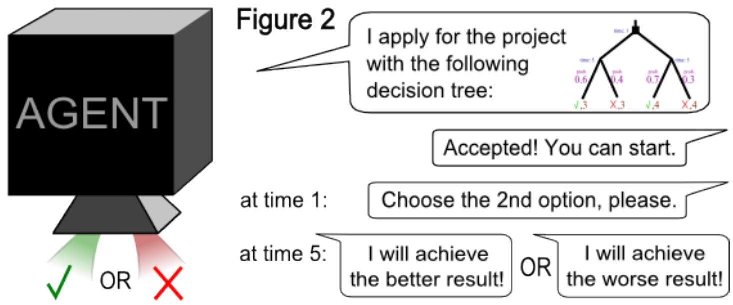


Figure 2

example, one agent cannot start building the roof until another completes the walls. Or they may have to share common resources, such as a loading area. We assume that the completion time of the walls and the sharing of a common resource are verifiable. In general, we assume that whenever a consequence of the execution of the workflow of an agent has influence on others, then these consequences are verifiable, meaning that the payments to the agents can depend on these verifiable consequences. The workflow of each agent is modeled by a dynamic process with hidden decisions and private stochastic feedbacks. Therefore, in order to maximize expected social welfare, each agent should make each of his hidden decisions depending on the hidden abilities and the previous private feedbacks of all players. Our goal in this paper is to incentivize them in this efficient joint behavior, even though they can lie about their abilities and their stochastic feedbacks, and they can choose selfish actions without the risk of being caught.

We begin with modeling the workflow of an agent, including his hidden actions and the private feedbacks he gets. We model a workflow as an arbitrary one-player stochastic game, or in other words, a two-player game between an agent and nature. The game has an absolute timing for the moves of the two players, and the game ends with a public consequence and a private cost.

A simple example for a workflow is described by the decision tree in Figure 1, as follows. At time point 1, the agent has to choose between two possible working processes. The left process has a total cost of 3; and with probabilities 0.6 and 0.4, the process finishes with a positive or negative consequence, respectively. The right process costs 4, but the probabilities are 0.7 and 0.3 for the positive and negative consequences, respectively. In this simple example, with both processes, the agent has only the prior probabilities for the consequences until time point 5, when the consequence become known to him.

The decision tree, the moves of the game and the cost of the work are all private information of the agent. In other words, each agent is a black box which can communicate with us and outputs a consequence at the end, but about whom nothing else is observable, including any prior distribution for his private two-player game. Figure 2 shows an example what an agent looks like to all other players during the execution of the project. For example, the agent could easily choose the first, cheaper process instead of the second one (as instructed) without the risk of being caught. Moreover, if, for example, the agent had no decision opportunity, but he had a cost of 1 and probability 0.5 of the positive consequence, then he would easily be able to play as described in Figure 2.

Now let us see the description of a very simplified version of the model. There is a central player called the principal, and some competing agents with private types describing their workflows. Contractible communication between them is available throughout the game. The principal is free to choose which agents to involve in the project, and the others leave the game. Then each of the chosen agents executes his own private workflow described by his decision tree. At the end, the principal gets the valuation of the entire project, which is a function of all workflow consequences. In addition, each agent pays the cost of the execution of his

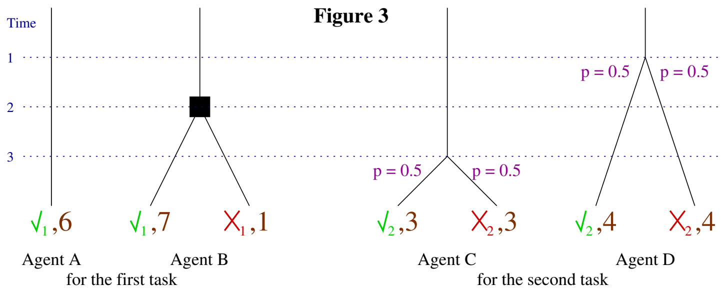


Figure 3

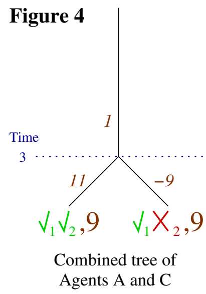


Figure 4

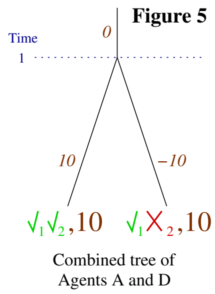


Figure 5

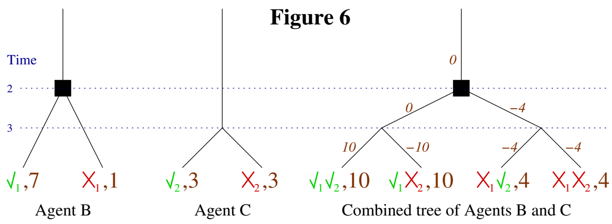


Figure 6

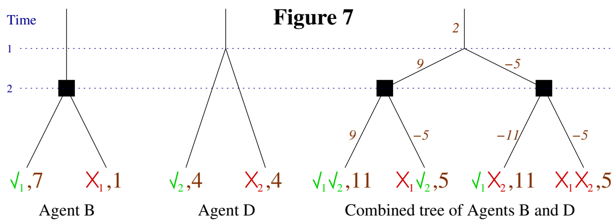


Figure 7

workflow, and the principal pays the agents depending on the achieved consequences and the communication history. All players are risk neutral, and the utilities are quasi-linear. We will see later that the influences of the agents on each other can be expressed by the joint valuation of the workflow consequences.

We emphasize that our main motivation is to avoid the typical global inefficiency of the project due to differences in incentives. With a reasonable tendering mechanism, and under reasonable competition, the expected utilities of the winning agents are not very high relative to the losses we want to avoid. Therefore, which mechanism lets the agents gain slightly less total profit is not that important. The real goal is to reduce social inefficiency.

1.1 Example 1 – The simplest interesting example

As a very simple example, consider the following two-task project. The principal must choose one agent per task, and each task i has two possible consequences: success ($\checkmark_i$) or failure ($\times_i$). If both tasks succeed, then the principal gets $v(\checkmark_1\checkmark_2) = 20$. But if either fails, then the consequence of the other task is irrelevant, and the principal receives $v(\checkmark_1\times_2) = v(\times_1\checkmark_2) = v(\times_1\times_2) = 0$.

Agents A and B apply for the first task, Agents C and D apply for the second task. Their decision trees are described in Figure 3. Agent A would complete the first task with a cost of 6. Agent B would complete the first task with a cost of 7, but he would have an option of quiting at time point 2 with total cost 1. Agent C would have a cost of 3, his consequence is either success ($\checkmark_2$) of failure ($\times_2$) of the second task, each has probability 1/2, and he observes this consequence at time point 3. Agent D has the same workflow but with cost of 4, and he observes his consequence at time point 1. (We omit denoting the probabilities, but all are 1/2 in our example.)

At the beginning, each agent is asked to report his private type. We show the execution of the game when all agents report their workflows truthfully, and they act truthfully and obediently throughout the game.

If the principal were to choose Agents A and C, then for a total cost of $6 + 3 = 9$, the principal would get $v(\checkmark_1\checkmark_2) = 20$ with probability 1/2. This would provide $1/2 \cdot 20 - 9 = 1$ expected social welfare. If the principal were to choose Agents A and D, then the total cost would be $6 + 4 = 10$, and the expected social welfare would be $1/2 \cdot 20 - 10 = 0$. If the principal were to choose Agents B and C, then the best strategy would be that Agent B should choose the left option, with a total cost of $7 + 3 = 10$, and the expected social welfare would be $1/2 \cdot 20 - 10 = 0$. However if the principal were to choose Agents B and D then the best strategy would be for Agent B to choose the left option if Agent D succeeds, or the right option if Agent D fails. This way, the expected total cost would be $(7+1)/2+4 = 8$, and the expected social welfare would be $1/2 \cdot 20 - 8 = 2$.

The expected social welfare can be calculated by a simple recursion as follows. Figures 4 and 5 and the "combined decision trees" in Figures 6 and 7 show the joint execution of the workflows of all pairs of agents for the two tasks. We call them combined decision trees. Such a tree describes all possible joint executions of the two workflows. For example, the path to the second leaf of the third tree in Figure 6 describes the execution that Agent B chooses the first option at time point 2, and then Agent C has the second chance event at time point 3. The leaf has the pair of the corresponding consequences: success of the first task and failure of the second task; and we have a total cost of $7 + 3 = 10$. The social welfare is the value of the pair of consequences (20 if both succeed, 0 otherwise) minus the total cost. For example, at this leaf, the social welfare would be $v(\checkmark_1 \times_2) - (7 + 3) = 0 - 10 = -10$; and at the first leaf in Figure 4, this would be $v(\checkmark_1 \checkmark_2) - (6 + 3) = 20 - 9 = 11$.

The state of the project is a point in the graph of the combined decision tree. This corresponds to the vector of the current subworkflows of the agents involved in the project, given their previous decisions and chance events. The value of a state of the project (indicated in brown italics) is defined to be the maximum expected social welfare assuming an efficient strategy profile. Alternatively, it can be defined recursively from the bottom to the top, as follows.

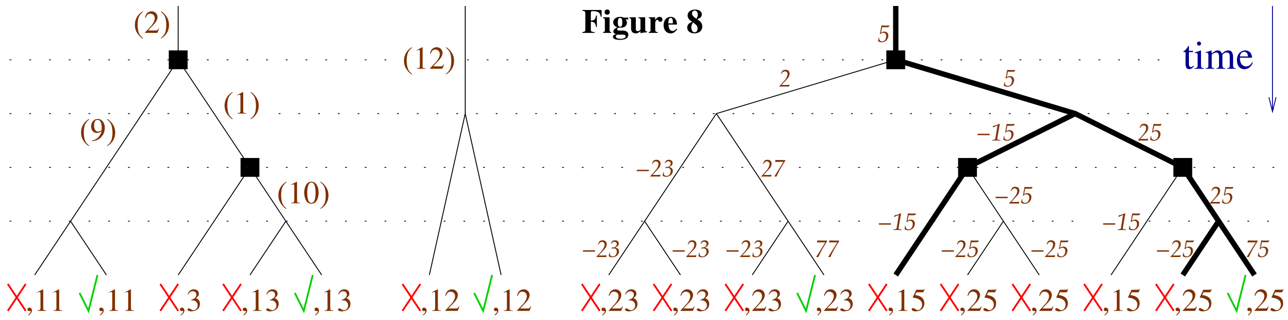
- The value of a leaf is calculated as the joint value of the consequences minus the sum of the costs.
- The values of two points lying on the same edge of the decision tree (with no decision or chance node between them) are the same.
- The value at/before a decision node is equal to the maximum of the values of the edges connected to the decision node from below.
- The value of a chance node is equal to the mean of the values of the edges connected to the chance node from below, weighted by their probabilities.

The value of the starting state is called the value of the pair (or subset, in general) of agents.

Once the values have been calculated, the project is executed in the following way. The principal chooses the pair (subset) of agents with the highest value and considers their joint reported workflow. At each decision node, the principal observes which of the branches have the highest value, and asks the corresponding agent to choose that branch. At each chance node, the corresponding agent reports the chance event, and the principal pays him the change in the value of the current reported state of the project, or in other words, the signed difference between the values of the states of the project after and before the chance event. At the end, each agent has to deliver the reported consequence (defined by his entire communication with the principal), and the principal pays him the reported cost at the leaf. Furthermore, each agent gets a second price compensation: the difference between the value of the chosen subset minus the highest value of the subsets excluding the agent.

In our example, the values of pairs are: $V(A, C) = 1$, $V(A, D) = 0$, $V(B, C) = 0$ and $V(B, D) = 2$, therefore, the principal chooses the pair (B, D). The best pair without B is (A, C), the best pair without D is also (A, C), therefore, both agents get second price compensations $V(B, D) - V(A, C) = 1$. Then Agent D reports his chance event at time point 1. If he succeeds, then he will get $9 - 2 = 7$ for this chance event, and Agent B will be asked

Figure 8



(a) decision tree of Agent A **(b)** Agent B **(c)** combined decision tree of Agents A and B

to choose the first branch. But if Agent D fails, then he gets $(-5) - 2 = -7$ (he pays 7) for this chance event, and Agent B will be asked to choose the second branch. In total, Agent B receives his reported cost plus his second price compensation, which is $7 + 1 = 8$ or $1 + 1 = 2$ payment depending on whether he was asked to choose the first or second branch, respectively. Agent D receives $4 \pm 7 + 1$, namely, he gets 12 or pays 2 depending on whether he succeeds or fails at his task, respectively. (We note that these high risks are only the properties of our simple and extreme example. Real-life games generally contain a large number of smaller risks, providing much smaller total risk. For example, you can imagine that we have hundreds of small independent identical projects of this kind, and the same agents play all of them. For further discussion of this topic, see Appendix B.4.)

We note that the calculation we used is only a definition and not a suggested algorithm.

In this article, we prove that the players are interested in similar truthful and obedient strategies, and in this case, the process works efficiently.

1.2 Example 2 – Evaluation of a pair of agents

Consider again a two-task project where the principal must assign one agent per task. Again, we consider only two possible consequences of the execution of each task: success (✓) or failure (✗). The project involves high risks but potentially high reward: if both tasks succeed, then the principal gets 100, but if either fails, then she receives 0.

Figure 8(a) shows the decision tree of Agent A, who could work on the first task, and Figure 8(b) shows the tree of Agent B, working on the second task. The probabilities at all chance nodes are $1/2$.

Initially, Agent A starts with a cost of 2. Next he chooses between the left and right options. If he chooses the left option, then his cost will be $2 + 9 = 11$ and his probability of success will be $1/2$. The right option will initially give him a total cost of $2 + 1 = 3$, but at a later point in time he will either be asked to quit and give up the task, or be asked to complete the task with a further cost of 10 and probability of success $1/2$. Agent B has a total cost of 12 and probability of success $1/2$. The timing of all decision nodes and chance events are shown in the two figures.

We emphasize that timing plays an important role because this determines what chance events of other agents are observed by the time point of each decision node. The cost on each edge is shown in parentheses, and their sum along a full path gives the cost at the leaf where the path ends.

The entire process starts with a number of agents applying for either of the two tasks by sending a type. The principal considers each pair of reported types for the two different tasks, evaluates the maximum expected utility with them, and chooses the pair with the highest utility. This example shows only the evaluation of one pair including what would happen if she were to accept this pair, and both would behave truthfully and obediently. The evaluation of the two applications in Figures 8(a) and 8(b) is described in Figure 8/c. It is constructed from working through the decision and chance nodes in chronological order. In this example, Agent A's top decision node occurs first, with two edges stemming from it. The next event is Agent B's chance node, which we plot at the end of each of our two edges. The two leaves of Agent B come from each of these two nodes, to meet the corresponding lower portions of Agent A's decision tree. For example, the path to the 5^{th} leaf from the left in Figure 8(c) describes the pair of the paths to the middle leaf of Figure 8(a) and the left leaf of Figure 8(b). In real terms, Agent A chooses the right option, then Agent B fails to complete the task, and then Agent A quits his task. At each leaf, ✓ denotes overall success, ✗ denotes overall failure (failure of at least one task), and the total cost of the corresponding path is numbered.

We calculate the values of the states of the project and the efficient joint strategy by the same method as described in Example 1. For Agent A, this means that the principal asks him – perhaps surprisingly – to work in the second way, namely, to make only preparations, and then, beyond the second price compensations, the principal either asks him to do nothing and he gets 3, or she asks him to try to complete the task, and he gets 15 and ± 50 for the risk, namely, he gets 65 if he succeeds, and he pays 35 if he fails. For Agent B, this means that he gets 11 and ± 20 for the risk, i.e., if he succeeds then he receives 31, but if he fails then he pays 9.

We note that the utility of the principal is always equal to the value of the starting state, because whenever the value of the current state of the project changes she pays this difference, and at the end she gets the value of the end-state.

We call this mechanism the "second price mechanism", but we consider a "first price mechanism" where we change the payment rule by excluding the second price compensation. Under first price mechanism, we expect from the agents reporting their types with costs increased by a constant. Applied to this example, in Figure 8(a) the number 2 shown at the root is not in fact a cost but a desired expected utility. Similarly 12 can express, say, cost 9 plus desired expected utility 3. The first price mechanism is not completely efficient (because not even a first price sealed bid is efficient), but we will show that this is collusion-resistant.

2 Comparison with existing models

First, we show a somewhat lossy translation of our model to the language used by Athey and Segal (2013) [1]. This version is not capable of showing all our results, it hides our main motivation and we omit some minor details here, but this version is better for a comparison of our second price mechanism with the related literature.

We have a set $A = \{1, 2, ..., I\}$ of agents, and there is a countable number of periods $t \in \mathbb{N} = \{0, 1, \dots\}$.

In period 0, each agent $i \in \{2, 3, ..., I\}$ privately observes his type $\theta_0^i \in \Theta^i$, and agent 1 publicly observes his type $\theta_0^1 \in \Theta^1$. The type vector $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta} = \prod_{i=1}^{I} \Theta^i$ is chosen arbitrarily, with no prior distribution. Then, each agent except agent 1 can report that he does not participate.

Each period $t \in \{1, 2, \dots\}$ consists of the following steps.

- Each agent $i \in A$ makes a private decision $x_t^i \in X^i$.
- A public decision $x_t^0 \in X^0$ is made.
- Each agent $i \in A$ privately observes his realized private state $\theta_t^i \in \Theta^i$.
- All agents observe a realized public state $\theta_t^0 \in \Theta^0$.

- A public consequence vector $\boldsymbol{c}_t \in C^A$ is announced.
- Each agent $i \in A$ is given a transfer $y_t^i \in \mathbb{R}$ with $\sum\limits_{i \in A} y_t^i = 0$. If agent $i \in \{1, 2, ..., I\}$ reported that he does not participate and $c_t^i = \emptyset$, then $y_t^i = 0$.

The utility of each agent $i \in A$ is given as a function of the sequences of his private states $(\theta_t^i)_{t=0}^{\infty}$, the public states $(\theta_t^0)_{t=0}^{\infty}$, his private decisions $(x_t^i)_{t=1}^{\infty}$, the public decisions $(x_t^0)_{t=1}^{\infty}$, the consequences $(\boldsymbol{c}_t)_{t=1}^{\infty}$ and monetary transfers $(y_t)_{t=1}^{\infty}$, as follows:

$$utility(i) = \sum_{t=1}^{\infty} \delta^t \Big(v^i(x_t^i, x_t^0, \theta_t^i, \theta_t^0, \boldsymbol{c}_t) + y_t^i \Big),$$

where $\delta \in (0, 1)$ is a discount factor.

The distribution of subsequent states of each agent $i \in A$ is governed by transition probability measure $\mu^i \colon X^{\{i,0\}} \times \Theta^{\{i,0\}} \times C^A \to \Delta(\Theta)$. Specifically, for any period $t \geq 0$, the next period state of $i \in A$ is a random variable $\tilde{\theta}_{t+1}^i$ distributed according to the probability measure $\mu^i(x_t^i, x_t^0, \theta_t^i, \theta_t^0, \boldsymbol{c}_t) \in \Delta(\Theta^i)$. Similarly, $\tilde{\theta}_{t+1}^0$ is distributed according to $\mu^0(x_t^0, \theta_t^0, \boldsymbol{c}_t) \in \Delta(\Theta^0)$.

Each consequence c_t^i is announced as a function of x_t^i, x_t^0, θ_t^i and θ_t^0.

We note that using consequences or public decisions are equivalent. On one hand, a consequence is equivalent to a public decision where all but the desired decision would provide a valuation of $-\infty$ for the agent. On the other hand, a public decision is equivalent to the same private decision which is included to the same-period consequence of the agent.

We will design a mechanism that determines monetary transfers depending on the public information, including the history of public reports of the agents. We will show that this is incentive-compatible in a surprisingly strong sense, and individually rational without offering free lunch.

2.1 Related literature

Our second price mechanism can be considered a dynamic extension of the Vickrey–Clarke–Groves mechanism (1961) [7], but in a much more complex form.

In our model, each agent receives private signals and makes private actions. However, there is also a publicly observable joint function of these signals and actions. This function value is called the workflow consequence of the agent, and this (alone) can have an effect on other agents.

For instance, in our model, we cannot differentiate between whether an agent (a) had bad luck, (b) were lazy (achieved worse consequence for less cost), or (c) had worse abilities (types) than he initially reported. However, our aim is to incentivize truthful strategies.

In our model, each agent receives private signals and makes private actions, but only some joint consequences of them can be publicly observed. For instance, in our model, we cannot differentiate between whether an agent (a) had bad luck, (b) were lazy (achieved worse consequence for less cost), or (c) had worse abilities (types) than he initially reported. However, our aim is to incentivize truthful strategies.

Similar questions in dynamic mechanism design was analyzed by Athey and Segal (2013) [1]. Their general setup with the unbalanced team mechanism is not really related to our result, because we do not deal with dependent private signals. But their setup with independent types is very similar to our setup. However, their "balanced team mechanism" is different from our "second price mechanism", and the equilibrium concepts are also different. Athey and Segal use perfect Bayesian equilibrium which is a weak concept, and we will show below that in their model, there can also exist inefficient perfect Bayesian equilibria which seem to be more plausible in practice. In contrast, we use a much stronger and much more convincing equilibrium concept.

Consider first the unbalanced team mechanism of Athey and Segal, which is roughly the following. The agents report all their private information, and according to these reports, the mechanism makes the socially efficient public decision, makes recommendations for the private decisions, and everybody receive the monetary transfer equal to the sum of the valuations of the other agents. They show that truthtelling is a perfect Bayesian equilibrium.

Consider the following setup with two agents. In each round, each agent receives a signal \$1000 or -\$1. Then they make a public decision "yes" or "no". If they choose "no", then both of their valuations are 0. But if they choose "yes", then their valuations equal the amount in the message.

Consider the case when an agent receives signal $-\$1$. If he reports \$1000 instead, then this costs him at most \$2, but this provides at least \$999 more utility to the other agent. Therefore, reporting \$1000 as long as the other agent also does so is another perfect Bayesian equilibrium in the infinite-horizon game.

This seems to be not only an abstract counterexample. Similar but less extreme phenomena seems to be very common, unless if the valuations are public information. Namely, whenever the report of an agent has a significant effect on his own reported valuation, then he may report higher valuation in order to build a long-term cooperation relationship with the others, even if his fake report causes some loss in social efficiency.

To a lesser degree, the same phenomenon may also occur under the balanced team mechanism, which is roughly the following. Whenever an agent i reports a new private signal, the player gets the monetary transfer equal to the expected change on the total expected valuations of all other players during the game. This (signed) monetary transfer is paid by the other agents who share this *equally*.

Consider the following setup with 101 agents including Alice, Bob playing a repeated game. In each round, there is a good to be allocated to one agent. In every round, the true valuation of Alice and Bob of the good are around \$1000, and the valuations of all other agents are clearly lower. In some rounds, Alice knows in advance that her valuation is \$1000; the valuation of Bob is either \$999 or \$1001 with probability 1/2 for each, and he gets to know which one just before the decision. If Bob receives the signal \$1001, but he reports \$999 instead, then Bob will get utility \$500 instead of $\$1001 - \$500 = \$501$, but Alice gets utility $\$1000 - \$500/100 = \$995$ instead of $\$500/100 = \$5$. Assume that Alice and Bob change role alternately. Then reporting \$999 as long as the other agent does so is another perfect Bayesian equilibrium which is better for Alice and Bob, but not socially efficient.

In contrast, in our mechanism, we do not share the transfer equally, but each agent $j \neq i$ pays to i the change of his own expected total valuation. Therefore, given the types of the agents, the expected utility of each agent with truthful strategy is fixed, even if all other agents collude.

There is one more important difference. They assume publicly known starting states as a parameter of the mechanism. This assumption goes far beyond the common prior assumption, namely, this means that the common prior is assumed to be verifiable by a court. At least, hard to find applications when the agents receive information dynamically, which the other players cannot completely observe, but the "precise *a priori* distribution" of the entire information flow is verifiable by a court. Especially because the probability distribution of the information flow is a function of the flow of private and public decisions. Moreover, the assumption of independent types means that the common prior must be independent across agents. This makes the assumption even less plausible. In contrast, in our model, all but one players have private types with no prior. (We think of the exceptional player as the owner of the project.)

Our mechanism also satisfies some important requirements: to be individually rational but avoids free riders. Furthermore, we present a version of our mechanism which is collusion-resistant, but which is less efficient in imperfect competition.

There is another related paper presented by Bergemann and Välimäki (2010) [2], who used an unbalanced dynamic mechanism for social allocations with independent private signals. Their setup is similar to the special case of our setup but without public and private decisions of the agents. They introduce a mechanism different both from our mechanism and from the mechanism of Athey and Segal, and they show that truthfulness is a perfect Bayesian equilibrium under the mechanism. This equilibrium, again, has similar weakness as in the result of Athey and Segal, as we show in the example below. In return, their mechanism satisfies an exit condition. The mechanism is roughly the following. The agents are asked to report all of their private information, then we make the socially efficient public decision, and each agent i pays the total expected decrement of the valuations of others caused by taking account of the preference of i.

Consider the following setup with at least three agents: Alice, Bob and others. In each round, there are two options: "yes" or "no", and the rounds are independent. Both Alice and Bob have a preference of about \$1000 for "yes", meaning that their valuations are \$1000 for "yes" and \$0 for "no". The other agents prefer "no", their total preference is about \$2000. The precise amounts will be private information, but their public *a priori* distributions are highly concentrated. Suppose that everybody report truthfully. If the total preference for "no" is the higher, then this will be the decision, and Alice and Bob will pay nothing. But if "yes" wins by a small amount x, then "yes" will be the decision and both agents pay their reported preference minus x, therefore, both of their utilities will be x. To sum up, the utilities of Alice and Bob are around 0 anyway.

Consider now what happens if Alice reports much more, say, \$3000 for "yes", compared with her truthful strategy. If "yes" won even with true reports, then there is no difference for Alice. If "no" won by x, then this misreport changes the decision to "yes", but Alice should pay her true preference plus x, therefore, she gets $-x$ utility instead of 0. This is still a marginal difference. However, in both cases, Bob gets his more favorable decision and he would not have to pay anything, therefore, his utility equals his valuation, which is about \$1000. Therefore, Alice is able to provide a much higher utility for Bob, for at most marginal losses in her own utility. This works vice versa, and if both of them report a much higher preference, then "yes" will be the decision and they will not need to pay anything. Therefore, in the infinite repeated game, it is very likely that Alice and Bob will report higher preferences as long as the other one also does so.

This kind of problem cannot occur in all situations, but still, one should be very careful when applying this mechanism. In contrast, we use a much stronger equilibrium concept, therefore, our mechanism never has such problems.

We will use some well-known techniques for our results. The revelation principle was introduced by Gibbard (1973) [6] and extended to the broader solution concept of Bayesian equilibrium by Dasgupta, Hammond and Maskin (1979) [4], Holmstrom (1977) [9], and Myerson (1979) [11]. It tells us that any equilibrium of rational communication strategies for the agents can be simulated by an equivalent incentive-compatible direct-revelation mechanism, where a trustworthy mediator maximally centralizes communication and makes honesty and obedience rational equilibrium strategies for the agents. Accordingly, we will achieve our goal by a stochastic dynamic direct mechanism.

Ex post Nash equilibrium was discussed as "uniform incentive compatibility" by Holmstrom and Myerson (1983) [8] and it is increasingly studied in game theory (see Kalai (2002) [10]) and is often used in mechanism design as a more robust solution concept (Cremer and McLean

(1985) [3], Dasgupta and Maskin (2000) [5], Perry and Reny (2002) [13]). Our (main) solution concept will be a kind of ex post Nash equilibrium which is ex post with respect to the private information of other players, and ex ante with respect to the future chance events. In order to prove this equilibrium, we will use the technique introduced by Myerson (1986) [12].

Our second price compensation will be analogous to the technique used in the Vickrey auction (1961) [15] and Groves mechanisms (1973) [7].

3 The model

We will use different notation than what we used in Section 2. For a full description of the notation, see Appendix C.

3.1 Interpretation

In contrast to the version in Section 2, this setup is designed in order to show the first price mechanism rather than the second price mechanism.

We consider a setting where a principal wants to get a particular project completed by some agents. We focus on the case when there are many agents and no agent is indispensable for the project. The principal can be considered as the owner of the project and the designer of the mechanism. Therefore, she has a publicly known strategy, and is not dealt with in our equilibrium concept.

The principal can negotiate with the agents about the payment rules, namely, how the payments depend on the achievements of the agents and on the messages they send to each other during the work. If an agent does not agree with the principal, then the agent leaves the game.

Each player has a private workflow. This is a dynamic stochastic game resulting in public consequences. Consequences are verifiable by a court. Each player has a joint valuation of the execution of his/her workflow and the consequences of the other players. For example, in Example 2 (Section 1.2), these concepts are the following.

Example 2	workflow	consequence	valuation function
principal	trivial: no decisions, no chance events	empty	$v(\checkmark_1\checkmark_2) = 100$ $v(\checkmark_1\times_2) = v(\times_1\checkmark_2) = v(\times_1\times_2) = 0$
Agent A	Figure 8(a) without costs at the leaves	success or failure of task 1	the negative of the cost at the leaf where his execution is finished
Agent B	Figure 8(b) without costs at the leaves	success or failure of task 2	the negative of the cost at the leaf where his execution is finished

Notice that whereas in our previous examples, an agent's type was simply his workflow including costs, in our model, each agent's type consists of (a) his workflow without costs and (b) his valuation function (which accounts for the costs).

For each player, we assume that (s)he has some basic information such as his/her own workflow, but we do not assume anything about how much this player knows about other players. Based on the idea of Myerson (1986) [12], we handle this in the following way. We define the model so that the players have extended (almost complete) information, and we are looking for an equilibrium $\boldsymbol{s}^*$ in which every player uses only his basic information. The reason is the following. Consider now the model when the players have less information but they still have their basic information. On one hand, $\boldsymbol{s}^*$ is still a valid strategy profile. On the other hand, no agent has an incentive to deviate, otherwise this would be a beneficial deviation with extended information, as well. Despite this argument is made for static Nash-equilibrium, this will also be valid for dynamic quasi-dominant Nash-equilibrium.

We require each player to be the first to know each chance event in his/her tree, namely, there is a time point when this player knows the result of the chance event but nobody else does. This is the only incompletion of information.

3.2 The game

We strongly recommend Section 3.1 for a proper understanding of the context of the game.

We deal with the following game, denoted by G.

Players. There is a player P called the **principal** and players $1, 2, ..., n$ called **agents**. Let $\mathcal{A}g = \{1, 2, ..., n\}$ and $\mathcal{P}l = \mathcal{A}g \cup \{P\}$. (Formally, nature is also a player in the sense that it makes actions. However, here we will use the word *player* only for agents and the principal.)

Each player $y \in \mathcal{P}l$ has a **workflow** d_y which is an arbitrary two-player game between y and nature. Each move of nature in d_y, called a **chance event** of y, is chosen from a probability distribution defined in d_y, and chosen independently from everything outside of d_y. Each move of each player should be made at a given absolute time point in $[0, 1]$. Denote the set of **executions** of d_y by $\mathcal{L}_y$. The set of possible consequences $\mathcal{C}$ is a union-closed set of sets including $\emptyset$ (that is, $\emptyset \in \mathcal{C}$ and $X, Y \in \mathcal{C} \Rightarrow X \cup Y \in \mathcal{C}$), and a **consequence** is assigned to each possible execution $l \in \mathcal{L}_y$. The actual execution of the workflow of player y is denoted by l_y, and its consequence is denoted by $c_y = c_y(l_y)$.

Each player $y \in \mathcal{P}l$ has a **valuation function** $v_y : \mathcal{L}_y \times \mathcal{C} \to \mathbb{R}$. We call the pair $\theta_y = (d_y, v_y)$ the **type** of player y.

The actions of the players and of nature are the following.

- At the beginning of the game (say, at time point -1), nature assigns a type θ_y to each player $y \in \mathcal{P}l$ in an arbitrary way. (Nondeterministic action, no prior.) Each agent observes his type privately.
- Each agent can send a (private but contractible) time-stamped instant message to the principal at each time point, and vice versa.
- At time point 0, the principal chooses a subset $\mathcal{A}cc \subset \mathcal{A}g$ of agents called **accepted agents**. $\mathcal{A}cc \cup \{P\}$ is called the set of **accepted players**.
- The workflows of all accepted players are executed during $[0, 1]$.
- At the end (at time point 1), the principal determines the **payment** vector $\boldsymbol{p} \in (\mathbb{R} \cup -\infty)^{\mathcal{A}cc}$.

The information (also called extended information[1]) of a player y at a time point consists of the following.

- The strictly previous actions of all players and nature (including the types).
- His/her own chance event at the current time point (if exists).

We denote the total consequence by $\boldsymbol{c} = \bigcup_{y \in \mathcal{A}cc \cup \{P\}} c_y$.

[1]Recall that (Section 3.1) the extended information is just an upper bound on the "real information". See also the basic information in Section 3.3.

The utility $\boldsymbol{u}$ of the players are defined as follows.

$$u_i = \begin{cases} v_i(l_i, \boldsymbol{c}) + p_i & \text{if } i \in \mathcal{A}cc \\ 0 & \text{if } i \in \mathcal{A}g - \mathcal{A}cc \end{cases} \tag{1}$$

$$u_P = v_P(l_P, \boldsymbol{c}) - \sum_{i \in \mathcal{A}cc} p_i \tag{2}$$

The subgame after nature assigns the type vector $\boldsymbol{\theta}$ to the players is denoted by $G(\boldsymbol{\theta})$. Most expressions will be functions of the types as nondeterministic parameters, but when there is no ambiguity, we will drop these parameters.

3.3 Goal

The **basic information** of each player is defined as the part (or formally, a function) of his/her information consisting of the following.

- his/her own type
- all messages (s)he received strictly previously
- whether he is accepted (only for agents, if the decision has already been made)
- his/her chance events up to and including the current time
- his/her previous actions
- at the end (at time point 1), the vector of consequences $\boldsymbol{c}_{\mathcal{P}l}$

We define a **simple strategy** of a player to be a strategy that assigns an action to his/her basic information.

Let E denote the expected value, which means (by default) the marginalization over the chance events. Note that $\mathrm{E}(u_i)$ is a function of the types and strategies of the players.

For any $S \subset \mathcal{P}l$, let $u_S = \sum_{y \in S} u_y$, and we call $u_{\mathcal{P}l}$ the **social welfare**. Clearly,

$$u_{\mathcal{P}l} \overset{(1)+(2)}{=} \sum_{y \in \mathcal{A}cc \cup P} v_y(l_y, \boldsymbol{c}). \tag{3}$$

For any subgame H of G, let the social value $V(H)$ be the maximum achievable expected social welfare in H, namely, (where $\boldsymbol{Str}$ denotes the set of all strategy profiles)

$$V(H) = \sup_{\boldsymbol{s} \in \boldsymbol{Str}^H} \mathrm{E}\big(u^H_{\mathcal{P}l}(\mathbf{s})\big). \tag{4}$$

Clearly,

$$\mathrm{E}(u_{\mathcal{P}l}) \le V(G). \tag{5}$$

Our main goal is to find a profile of simple strategies $\boldsymbol{s}^*$ satisfying

- *Efficiency:* $\mathrm{E}\big(u_{\mathcal{P}l}(\boldsymbol{s}^*)\big) = V(G)$
- *Incentive-compatibility:* $\boldsymbol{s}^*$ is a dynamic quasi-dominant Nash-equilibrium (see Section 5.2)
- *Collusion-resistance* (see Section 7.1)
- *Individual rationality for each agent:* $\forall i \in \mathcal{A}g,\ \mathrm{E}\big(u_i(\boldsymbol{s}^*)\big) \ge 0$
- *Offers no free lunch:* $\forall i \in \mathcal{A}g,$ if $\theta_i = 0$, then $\mathrm{E}\big(u_i(\boldsymbol{s}^*)\big) = 0$

Type $\theta_i = 0$ denotes the empty workflow with $v_i \equiv 0$. The empty workflow means that its only execution is $l_\emptyset$, and $c_i(l_\emptyset) = \emptyset$.

3.4 Equivalent extensions of the model

Assume that we have a mechanism that satisfies some of the goals including incentive-compatibility. We show that this mechanism satisfies the same goals in the generalizations of the model below.

First, notice that all results remain valid if we extend the basic information of the players.

Now, suppose that some parts (or functions) of a consequence of a player may be determined at an earlier stage of the execution. If these are publicly announced at that stage, then this just extends the basic information of others, which keeps the results valid.

Let us assume that all consequences include their time points. Consider now the extension of the model when a consequence of an agent i can affect the action set of another agent j. This can be handled by setting $v_j(l_j, \boldsymbol{c}) = -\infty$ for any execution l_j and consequence c_i where j chose an action outside of the set.

Now we can see, for example, how this model includes precedences, e.g. when an agent building the roof of a house cannot start until other agents finish building the walls.

A consequence of an agent may even enforce private decisions of others, and these decisions can affect the probabilities of his chance events. Therefore, this is again just an equivalent extension if consequences can affect probability distributions of chance events of others.

Furthermore, notice that extending the model with public decisions and public chance events can be handled by extending the decision tree of the principal with publicly observable chance events and decisions.

Finally, suppose that some agents can continue the game even after a rejection. The restriction is he must provide consequence $\emptyset$. We could get this extension by applying the mechanism as if the valuation function of the agents were increased by a large constant. However it will be easier just to check that the proofs extend to in this case. However, we will not consider this extension under the (not completely efficient) first price mechanism, for multiple reasons.

4 Mechanisms

A **mechanism** is identified by a simple strategy of the principal. This strategy assigns the following actions to her basic information:

- her communication;
- her choice of the set of accepted agents $\mathcal{A}cc$;
- her moves (in her workflow);
- her choice of the payment vector $\boldsymbol{p}$.

4.1 The first price mechanism

The first price mechanism is essentially the following. The principal asks the agents to report all their basic information throughout the game. The principal believes these reports, and she always makes and asks the agents to make the particular moves in order to maximize expected social welfare. The principal pays the reported cost of each agent, plus whenever a reported chance event of an agent modifies the expected social welfare, the principal pays him this (signed) difference.

More formally, the agents should use the communication protocol below. First, each agent $i \in \mathcal{A}g$ sends a type to the principal at time point -1. We call this message his **application** $\hat{\theta}_i = (\hat{d}_i, \hat{v}_i)$ (which can be different from his actual type θ_i). Also, we use the notation $\hat{\theta}_P = \theta_P$,

$\hat{d}_P = d_P$, $\hat{v}_P = v_P$ (as the principal knows her own type). Then, the principal considers a strategy profile $\hat{\boldsymbol{s}}$ of the **reported game** $G(\hat{\boldsymbol{\theta}})$ which maximizes $\mathrm{E}\big(u^{G(\hat{\boldsymbol{\theta}})}(\mathcal{P}l)\big)$. In detail, she chooses an

$$\hat{\boldsymbol{s}} \in \mathop{\arg\max}_{\boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}^{G(\hat{\boldsymbol{\theta}})}} \mathrm{E}\big(u_{\mathcal{P}l}^{G(\hat{\boldsymbol{\theta}})}(\mathbf{s})\big), \tag{6}$$

- and she chooses the same set $\mathcal{A}cc$ as what she would choose by $\hat{\boldsymbol{s}}$.
- The principal makes the moves in her own tree corresponding to $\hat{s}_P$, and she considers her own chance event in $G(\hat{\boldsymbol{\theta}})$ to be the same as her real chance event in G.
- Whenever an agent $i \in \mathcal{A}cc$ reaches a chance node in $G(\hat{\boldsymbol{\theta}})$, the principal requires him to report to her a chance event, and she considers the event according to this report to be the chance event in $G(\hat{\boldsymbol{\theta}})$. For any two chance events, we consider them to occur at distinct points in time. We resolve concurrences arbitrarily.
- Whenever an agent $i \in \mathcal{A}cc$ is about to make a move in $G(\hat{\boldsymbol{\theta}})$, the principal sends him the move corresponding to $\hat{s}_i$, and she considers this to be his move in $G(\hat{\boldsymbol{\theta}})$.
- In the end, the principal gets a reported execution $\hat{l}_y$ with a reported consequence $\hat{c}_y = \hat{c}_y(\hat{l}_y)$ of each $y \in \mathcal{A}cc \cup P$. For an agent $i \in \mathcal{A}cc$, if $c_i \neq \hat{c}_i(\hat{l}_i)$, namely, the reported and the actual consequences are different, then $p_i = -\infty$. Otherwise, denote the set of the chance events during the reported execution of $\hat{d}_i$ in $G(\hat{\boldsymbol{\theta}})$ by X_i, and for each $\chi \in X_i$, denote the states of the project preceding χ by T^χ and succeeding χ by T^χ_+. Let

$$\delta(\chi) = V(T^\chi_+) - V(T^\chi). \tag{7}$$

  Then the principal chooses the following payment to each agent i.

$$p_i = -\hat{v}_i(\hat{l}_i, \boldsymbol{c}) + \sum_{\chi \in X_i} \delta(\chi). \tag{8}$$

We defined $p_i = -\infty$ at $c_i \neq \hat{c}_i(\hat{l}_i)$ to express that the agents cannot lie about their consequences. Namely, the report of an agent about his type and the execution of his workflow may be completely different from the reality, but it must match with the observable reality. In other words, even if an agent lies, he must be prepared to explain what happened, how he has achieved his consequence. One way to guarantee this is the following. We add a chance node x just before each leaf l, and we create branches from x ending at a new leaf with each possible consequence. In the valuation function, we do not care about this last step of the execution. The agent assigns probabilities 1 to the branch going to the leaf with the original consequence, and 0 to all other branches. If an agent does so, and at this chance node, he reports the event corresponding to the real consequence, then $\hat{c}_i(\hat{l}_i)$ is always the same as c_i. From now on, we assume that each agent i uses a strategy that guarantees $\hat{c}_i(\hat{l}_i) = c_i$.

We note that the "value of a state of the project" in Examples 1 and 2 (Sections 1.1 and 1.2) corresponds to $V(T)$ here.

4.2 The second price mechanism

We define the **joint value of types** $\boldsymbol{\theta'}$ by $V(\boldsymbol{\theta'}) = V\big(G(\boldsymbol{\theta'})\big)$. The **value of an application** $\hat{\theta}_i$ is the reported marginal contribution

$$v_i^+ = v_i^+(\hat{\boldsymbol{\theta}}) = V(\hat{\boldsymbol{\theta}}) - V(\hat{\boldsymbol{\theta}}_{-i}). \tag{9}$$

The second price mechanism is the same as the first price mechanism except that the principal pays v_i^+ more to each agent $i \in \mathcal{A}cc$, namely,

$$p_i^{2nd} = p_i + v_i^+(\hat{\boldsymbol{\theta}}) = v_i^+(\hat{\boldsymbol{\theta}}) - \hat{v}_i(\hat{l}_i, \boldsymbol{c}) + \sum_{\chi \in X_i} \delta(\chi).$$

(To avoid ambiguity of notations, we denote the payment vector $\boldsymbol{p}$ under the second price mechanism by p^{2nd}.)

4.3 Definitions and notations

Application 0 means the offer for doing nothing for free. This application provides utility 0 to the agent under both mechanisms, which implies the individual rationality. Therefore, our further goals are the efficiency, incentive-compatibility with collusion resistance, and that the mechanism offers no free lunch. We identify the starting action of each agent i with $\hat{\theta}_i$.

Let G/s_P denote the game G given the strategy of principal, so that the principal is no longer considered a player in G/s_P. Let m_1 and m_2 denote the first and the second price mechanisms, respectively. Let $M = G/m_1$ be the default game and all notions defined in G are used correspondingly in M, e.g. s refers to $\boldsymbol{s}_{\mathcal{A}g}$. Given s, the execution of the games $M = G/m_1$ and G/m_2 are the same; the only difference is the utility function. Therefore, we can define the second price utilities in M, and we denote it by u^{2nd}. This means that

$$\forall i \in \mathcal{A}g: \qquad u_i^{2nd} = u_i + v_i^+(\hat{\boldsymbol{\theta}}), \tag{10}$$

$$u_P^{2nd} = u_P - \sum_{i \in \mathcal{A}g} v_i^+(\hat{\boldsymbol{\theta}}). \tag{11}$$

We define the **cost price strategy** cp_i of an agent $i \in \mathcal{A}g$ as follows. He applies with his real type, and then he always makes the move which the principal asks him, and at each chance node, he sends her the true chance event. (We do not call it truthful strategy, because we do not want to suggest that this is the desired strategy.)

5 Efficiency and incentive-compatibility under the second price mechanism

First, we prove that the second price mechanism achieves our goals (Section 3.3) with Nash-equilibrium. This is not the adequate equilibrium concept for our dynamic game, but this describes the essence of the proof, and this is much easier to understand. Next, we improve the results considering the dynamics.

5.1 Proofs of Nash-equilibrium, efficiency, individual rationality, and no free lunch

In view of Examples 1.1 and 1.2, the next five lemmas and Proposition 6 are easy to prove. In paricular, Lemmas 1 and 2 show that the values of the states of the project can be calculated in the same way as in Examples 1.1 and 1.2. The main idea of this section is in Theorem 7.

Lemma 1. *Denote the state of the project preceding a move of a player by T, and all possible states of the project succeeding the different moves by $T_1, T_2, ..., T_m$. Then*

$$V(T) = \max_k V(T_k). \tag{12}$$

Proof. In the subgame T, the strategy profile of the players consists of a move $k \in \{1, 2, ..., m\}$ and a strategy profile in T_k. Therefore,

$$V(T) \overset{(4)}{=} \max_{\boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}^T} \mathrm{E}\big(u_{\mathcal{P}l}^T(\mathbf{s})\big) = \max_k \max_{\boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}^{T_k}} \mathrm{E}\big(u_{\mathcal{P}l}^{T_k}(\mathbf{s})\big) \overset{(4)}{=} \max_k V(T_k). \qquad \square$$

Lemma 2. *Denote the state of the project at a chance node of a player by T, and all possible states of the project succeeding the chance event by $T_1, T_2, ..., T_m$. Let $w_1, w_2, ..., w_m$ denote the corresponding probabilities. Then $V(T) = \sum w_k \cdot V(T_k)$, or equivalently,*

$$\sum_{k=1}^{m} w_k \Big(V(T_k) - V(T)\Big) = 0. \tag{13}$$

Proof. There is a natural correspondence between $\boldsymbol{\mathcal{S}tr}^T$ and $\overset{m}{\underset{k=1}{\times}} \boldsymbol{\mathcal{S}tr}^{T_k}$, namely, each strategy profile in T consists of what the players would do after the different chance events. Thus,

$$V(T) \overset{(4)}{=} \max_{\boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}^T} \mathrm{E}\big(u_{\mathcal{P}l}^T(\mathbf{s})\big) = \max_{\boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}^T} \sum_{k=1}^{m} w_k \cdot \mathrm{E}\big(u_{\mathcal{P}l}^{T_k}(\mathbf{s})\big) = \sum_{k=1}^{m} w_k \cdot \max_{\boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}^{T_k}} \mathrm{E}\big(u_{\mathcal{P}l}^{T_k}(\mathbf{s})\big) \overset{(4)}{=} \sum_{k=1}^{m} w_k \cdot V(T_k).$$

$\square$

Lemma 3. *If the chance nodes of a player $y \in \mathcal{P}l$ correspond to his/her chance nodes in $M(\hat{\boldsymbol{\theta}})$ with the same probabilities, then*

$$\mathrm{E}\Big(\sum_{\chi \in X_y} \delta(\chi)\Big) = 0. \tag{14}$$

Proof. Using the notions in Lemma 2, $\mathrm{E}\big(\delta(\chi)\big) \overset{(7)}{=} \sum w_i \big(V(T_i) - V(T)\big) \overset{(13)}{=} 0$. Consequently, $\sum \delta(\chi)$, summing on all past chance events χ of $y \in \mathcal{P}l$, is a martingale, and therefore, the expected values of the sums at the end (left hand side) and at the beginning (right hand side) are the same. $\square$

The following lemma shows that the expected utility of the principal depends only on the applications, no matter how the agents behave afterwards. (If the principal has no chance event, then no expectation is required.) Recall that u_y is the utility with the first price payments, not with the second price payments.

Lemma 4.

$$\mathrm{E}(u_P) = V(\hat{\boldsymbol{\theta}}) \tag{15}$$

Proof. Denote the current reported state of the project by T, the reached reported end-state by N, denote the sequence of the reported chance events of $y \in \mathcal{P}l$ until T by $X_y(T)$, and let $X(T) = \bigcup_{y \in \mathcal{P}l} X_y(T)$. Then $V(T) - \sum_{\chi \in X(T)} \delta(\chi)$ is invariant during the game because of (7) and (12). Therefore,

$$V(\hat{\boldsymbol{\theta}}) = V\big(G(\hat{\boldsymbol{\theta}})\big) = V\big(G(\hat{\boldsymbol{\theta}})\big) - \sum_{\chi \in X(M(\hat{\boldsymbol{\theta}}))} \delta(\chi) = V(N) - \sum_{\chi \in X(N)} \delta(\chi)$$

$$\overset{(4)(3)}{=} \mathrm{E}\Big(\sum_{y \in \mathcal{A}cc \cup P} \hat{v}_y(\hat{l}_y, \boldsymbol{c}) - \sum_{y \in \mathcal{A}cc \cup P} \sum_{\chi \in X_y} \delta(\chi)\Big) = \mathrm{E}\big(\hat{v}_P(\hat{l}_P, \boldsymbol{c})\big) +$$

$$+ \sum_{i \in \mathcal{A}cc} \mathrm{E}\Big(\hat{v}_i(\hat{l}_i, \boldsymbol{c}) - \sum_{\chi \in X_i} \delta(\chi)\Big) - \sum_{\chi \in X_P} \delta(\chi) \overset{(8)(14)}{=} \mathrm{E}\Big(v_P(l_P, \boldsymbol{c}) - \sum_{i \in \mathcal{A}cc} p_i\Big) \overset{(2)}{=} \mathrm{E}(u_P). \qquad \square$$

Lemma 5. *The expected utility of each agent who uses cost price strategy is 0, no matter how the other agents behave. Formally,*

$$\mathrm{E}\big(u_i(cp_i)\big) = 0 \tag{16}$$

Proof. If $i \notin \mathcal{A}cc$, then $u_i \overset{(1)}{=} 0$; and if $i \in \mathcal{A}cc$, then

$$\mathrm{E}(u_i) \overset{(1)}{=} \mathrm{E}\Big(v_i(l_i, \boldsymbol{c}) + p_i\Big) \overset{(8)}{=} \mathrm{E}\Big(v_i(l_i, \boldsymbol{c}) - \hat{v}_i(\hat{l}_i, \boldsymbol{c}) + \sum_{\chi \in X_i} \delta(\chi)\Big) \overset{(14)}{=} \mathrm{E}\Big(v_i(l_i, \boldsymbol{c}) - \hat{v}_i(\hat{l}_i, \boldsymbol{c})\Big) = 0. \quad \square$$

The following proposition shows the efficiency of the cost price strategy profile.

Proposition 6.

$$\mathrm{E}\big(u_{\mathcal{P}l}(\boldsymbol{cp})\big) = V(G). \tag{17}$$

Proof. If $s = \boldsymbol{cp}$, then $\hat{\theta} = \boldsymbol{\theta}$, therefore,

$$\mathrm{E}(u_{\mathcal{P}l}) = \mathrm{E}(u_P) + \sum_{i \in \mathcal{A}g} \mathrm{E}(u_i) \overset{(15)(16)}{=} V(\hat{\boldsymbol{\theta}}) = V(\boldsymbol{\theta}) = V(G). \quad \square$$

Theorem 7. *The cost price strategy profile $\boldsymbol{cp} = \boldsymbol{cp}_{\mathcal{A}g}$ is a Nash-equilibrium under the second price mechanism.*

Proof. Notice that

$$\mathrm{E}(u_P) - v_i^+ \overset{(15)(9)}{=} V(\hat{\boldsymbol{\theta}}) - \big(V(\hat{\boldsymbol{\theta}}) - V(\hat{\boldsymbol{\theta}}_{-i})\big) = V(\hat{\boldsymbol{\theta}}_{-i}). \tag{18}$$

We show that the cost price strategy profile $\boldsymbol{cp}$ is a Nash-equilibrium, because

$$\begin{aligned}
\mathrm{E}(u_i^{2nd})\big(\boldsymbol{\theta}, (s_i, \boldsymbol{cp}_{-i})\big) &\overset{(10)}{=} \big(\mathrm{E}(u_i) + v_i^+\big)\big(\boldsymbol{\theta}, (s_i, \boldsymbol{cp}_{-i})\big) \\
&= \Big(\mathrm{E}(u_{\mathcal{P}l}) - \sum_{j \in \mathcal{A}g - i} \mathrm{E}(u_j) - \mathrm{E}(u_P) + v_i^+\Big)\big(\boldsymbol{\theta}, (s_i, \boldsymbol{cp}_{-i})\big) \\
&\overset{(5)(16)(18)}{\leq} V(\boldsymbol{\theta}) - \sum_{j \in \mathcal{A}g - i} 0 - V(\hat{\boldsymbol{\theta}}_{-i})\big(\boldsymbol{\theta}, (s_i, \boldsymbol{cp}_{-i})\big) = V(\boldsymbol{\theta}) - V(\boldsymbol{\theta}_{-i}),
\end{aligned}$$

with equation if $s_i = cp_i$. Therefore, for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$,

$$\mathrm{E}(u_i^{2nd})\big(\boldsymbol{\theta}, (s_i, \boldsymbol{cp}_{-i})\big) \leq V(\boldsymbol{\theta}) - V(\boldsymbol{\theta}_{-i}) = \mathrm{E}(u_i^{2nd})(\boldsymbol{\theta}, \boldsymbol{cp}). \quad \square$$

The following lemma implies that the second price mechanism offers no free lunch.

Lemma 8. *If $\theta_i = 0$ and $s = \boldsymbol{cp}$, then $u_i = u_i^{2nd} = 0$.*

Proof. If $\theta_i = 0$, then $\hat{\theta}_i(\boldsymbol{cp}) = 0$, therefore, whether or not i is accepted, $u_i = 0$. Furthermore, $V(\boldsymbol{\theta}) = V(\boldsymbol{\theta}_{-i})$, therefore,

$$u_i^{2nd} \overset{(10)}{=} u_i + v_i^+(\hat{\boldsymbol{\theta}}) \overset{(9)}{=} 0 + V(\boldsymbol{\theta}) - V(\boldsymbol{\theta}_{-i}) = 0. \quad \square$$

Summarizing the results, the second price mechanism is individually rational, offers no free lunch, and the cost price strategy profile is a Nash-equilibrium and maximizes social welfare.

5.2 The dynamic quasi-dominant Nash-equilibrium

To achieve the original goals, we only need to improve Theorem 7 to a dynamic version. First, we define the dynamic quasi-dominant Nash-equilibrium which we believe to be a novel concept designed especially to catch the incentive compatibility we have.

Whether an equilibrium concept is meaningful is not a mathematical question. However, we try to provide a justification as close to a mathematical proof as possible. For example, these justifications provide sketches of proofs that in the Bayesian version of the model (see Section A.4), all perfect Bayesian equilibria are efficient. Starting with a very special case, we arrive at the dynamic quasi-dominant Nash-equilibrium and its justification in several steps.

Step 1. Consider the following situation. Players A and B play an arbitrary deterministic *dynamic* game with complete information. Suppose that there is a strategy profile $\boldsymbol{s}^* \in \boldsymbol{\mathcal{S}tr}$ satisfying the following.

$$\forall s_B \in \mathcal{S}tr_B: \qquad u_A(s_A^*, s_B) \ge 1 \tag{19}$$

$$\forall s_A \in \mathcal{S}tr_A: \qquad u_B(s_A, s_B^*) \ge 1 \tag{20}$$

$$\forall \boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}: \qquad u_A(\mathbf{s}) + u_B(\mathbf{s}) \le 2 \tag{21}$$

Then $\boldsymbol{s}^*$ is an equilibrium.

Justification. In an arbitrary two-player game, if either player can guarantee himself utility 1, and even with collusion, they cannot get more utility in total, then both players will use the strategy guaranteeing utility 1.

More formally, (19) shows that A can get utility at least 1, therefore, if she is selfish and rational, then she will get expected utility at least 1. Comparing this with (21), this shows that B has no hope of getting expected utility more than $2 - 1 = 1$. But (20) shows that s_B^* guarantees him 1. Therefore, B has no incentive to deviate from s_B^*. And the same argument holds for A, as well. Therefore, we can rightfully say that $\boldsymbol{s}^*$ is an equilibrium. □

Note that this is already a new equilibrium concept. $\boldsymbol{s}^*$ is not a subgame-perfect equilibrium, because each player might miss the opportunity to completely utilize when the other player makes a bad move. As a simple counterexample, consider the following game. Player A has to choose his utility u_A from $[0, 1]$, then player B, after observing u_A, has to choose his utility u_B from $[0, 2 - u_A]$. Then the strategy profile of choosing 1 by both players satisfies all (19), (20) and (21), but it is not subgame-perfect: if agent A chose e.g. 0.6, then the best choice for agent B would be 1.4, but he chooses 1 instead.

(It is true that under some compactness condition, there exists a subgame-perfect Nash-equilibrium by which the players make the same actions as under $\boldsymbol{s}^*$, and all subgame-perfect Nash-equilibria provide the same utilities. We do not continue the comparison between our equilibrium and the subgame-perfect Nash-equilibrium concepts, partially because later (from Step 5), we should compare our concept with more and more difficult extensions of subgame-perfectness, and partially because we try to show that our concept is a very strong kind of equilibrium, much stronger than just a subgame-perfect Nash-equilibrium.)

Step 2. There is a set of players N playing a deterministic dynamic game with complete information. Suppose that there is a strategy profile $\boldsymbol{s}^* \in \boldsymbol{\mathcal{S}tr}$ satisfying the following.

$$\forall i \in N,\ \forall \boldsymbol{s}_{-i} \in \boldsymbol{\mathcal{S}tr}_{-i}: \qquad u_i\big((s_i^*, \boldsymbol{s}_{-i})\big) \ge 1 \tag{22}$$

$$\forall \boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}: \qquad \sum_{i \in N} u_i(\mathbf{s}) \le |N| \tag{23}$$

Then $\boldsymbol{s}^*$ is an equilibrium.

Justification. (22) implies that each player can get utility at least 1. On the other hand, each player has no hope of getting more expected utility than the maximum possible total utility of all players minus the sum of the guaranteed utilities of the other players, which is $|N| - (|N| - 1) \cdot 1 = 1$, according to (23) and (22). Therefore, each player $i \in N$ has no hope of getting more expected utility than his utility guaranteed by s_i^*. Hence, we can rightfully say that $\boldsymbol{s}^*$ is an equilibrium. □

Step 3. There is a set of players N playing a deterministic dynamic game with complete information. Suppose that there is a strategy profile $\boldsymbol{s}^* \in \boldsymbol{\mathcal{S}tr}$ satisfying the following.

$$\forall i \in N,\ \forall \boldsymbol{s}_{-i} \in \boldsymbol{\mathcal{S}tr}_{-i}\colon \qquad u_i\big((s_i^*, \boldsymbol{s}_{-i})\big) \geq u_i(\boldsymbol{s}^*) \tag{24}$$

$$\forall \boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}\colon \qquad \sum_{i \in N} u_i(\mathbf{s}) \leq \sum_{i \in N} u_i(\boldsymbol{s}^*) \tag{25}$$

Then $\boldsymbol{s}^*$ is an equilibrium.

Justification. (24) implies that each player $i \in N$ can get utility at least $u_i(\boldsymbol{s}^*)$. (We say that $u_i(\boldsymbol{s}^*)$ is *guaranteed* for i.) Each player $i \in N$ has no hope of getting more expected utility than the maximum possible total utility of all players minus the sum of the guaranteed utilities of the other players. Therefore, i has no hope of getting more expected utility than

$$\sup_{\boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}} \sum_{j \in N} u_j(\mathbf{s}) - \sum_{j \in N-i} u_j(\boldsymbol{s}^*) \overset{(25)}{=} \sum_{j \in N} u_j(\boldsymbol{s}^*) - \sum_{j \in N-i} u_j(\boldsymbol{s}^*) = u_i(\boldsymbol{s}^*).$$

Consequently, each player $i \in N$ has no incentive to deviate from s_i^*, and therefore, we can rightfully say that $\boldsymbol{s}^*$ is an equilibrium. □

Step 4. There is a set of players N playing a deterministic dynamic game with complete information. At the same initial time point, the players do simultaneous actions denoted by $a_i = a_i(s_i) \in \mathcal{A}_i$ for each player $i \in N$. Suppose that there are functions $f_i : \boldsymbol{\mathcal{A}}_{-i} \to \mathbb{R}$ for all $i \in N$, and a strategy profile $\boldsymbol{s}^* \in \boldsymbol{\mathcal{S}tr}$ with $a_i^* = a_i(s_i^*)$ satisfying the following.

$$\forall i \in N,\ \forall \boldsymbol{s}_{-i} \in \boldsymbol{\mathcal{S}tr}_{-i}\colon \qquad u_i\big((s_i^*, \boldsymbol{s}_{-i})\big) \geq f_i\big(\boldsymbol{a}_{-i}(\boldsymbol{s}_{-i})\big) \tag{26}$$

$$\forall \boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}\colon \qquad \sum_{i \in N} u_i(\mathbf{s}) \leq \sum_{i \in N} f_i(\boldsymbol{a}_{-i}^*) \tag{27}$$

$$\forall i \in N,\ \forall a_i \in \mathcal{A}_i\colon \qquad \sum_{j \in N-i} f_j\big((a_i, \boldsymbol{a}_{-i-j}^*)\big) \geq \sum_{j \in N-i} f_j(\boldsymbol{a}_{-j}^*) \tag{28}$$

Then $\boldsymbol{s}^*$ is an equilibrium.

Justification. Each player $i \in N$ has no influence on $\boldsymbol{a}_{-i}$. Therefore, $f_i(\boldsymbol{a}_{-i})$ is independent of s_i, and (26) implies that i can get at least $f_i(\boldsymbol{a}_{-i})$ utility. Each player $i \in N$ has no hope of getting more expected utility than the maximum possible total utility of all players minus the sum of the guaranteed utilities of the other players. Therefore, if the players other than i use $\boldsymbol{a}_{-i}^*$, then i has no hope of getting more expected utility than

$$\sup_{\boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}} \sum_{j \in N} u_j(\mathbf{s}) - \inf_{a_i \in \mathcal{A}_i} \sum_{j \in N-i} f_j\big((a_i, \boldsymbol{a}_{-i-j}^*)\big) \overset{(27)(28)}{=} \sum_{j \in N} f_j(\boldsymbol{a}_{-j}^*) - \sum_{j \in N-i} f_j(\boldsymbol{a}_{-j}^*) = f_i(\boldsymbol{a}_{-i}^*).$$

Consequently, each player $i \in N$ has no incentive to deviate from s_i^*, and therefore, we can rightfully say that $\boldsymbol{s}^*$ is an equilibrium. □

Step 5. There is a set of players N playing a dynamic game with complete information. The game is deterministic except that at the very beginning, nature chooses a type $\theta_i \in \Theta_i$ for each player $i \in N$, as a nondeterministic action with no prior probabilities. Then at the same time point, the players do simultaneous actions, denoted by $a_i = a_i(\boldsymbol{\theta}, s_i) \in \mathcal{A}_i$ for each player $i \in N$. Suppose that there are functions $f_i : \Theta_i \times \boldsymbol{\mathcal{A}}_{-i} \to \mathbb{R}$ for all $i \in N$, and a strategy profile $\boldsymbol{s}^* \in \boldsymbol{\mathcal{S}tr}$ with $a_i^* = a_i(s_i^*)$, satisfying the following.

$$\forall \boldsymbol{\theta} \in \boldsymbol{\Theta},\ \forall i \in N,\ \forall \boldsymbol{s}_{-i} \in \boldsymbol{\mathcal{S}tr}_{-i}\colon \qquad u_i\big(\boldsymbol{\theta}, (s_i^*, \boldsymbol{s}_{-i})\big) \geq f_i\big(\theta_i, \boldsymbol{a}_{-i}(\boldsymbol{\theta}, \boldsymbol{s}_{-i})\big) \tag{29}$$

$$\forall \boldsymbol{\theta} \in \boldsymbol{\Theta},\ \forall \boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}\colon \qquad \sum_{i \in N} u_i(\boldsymbol{\theta}, \mathbf{s}) \leq \sum_{i \in N} f_i(\theta_i, \boldsymbol{a}_{-i}^*) \tag{30}$$

$$\forall \boldsymbol{\theta} \in \boldsymbol{\Theta},\ \forall i \in N,\ \forall a_i \in \mathcal{A}_i\colon \qquad \sum_{j \in N-i} f_j\big(\theta_j, (a_i, \boldsymbol{a}_{-i-j}^*)\big) \geq \sum_{j \in N-i} f_j(\theta_j, \boldsymbol{a}_{-j}^*) \tag{31}$$

Then $\boldsymbol{s}^*$ is an equilibrium.

Justification. In short, we can use Step 4 for the subgame after the move of nature. Or, we can essentially repeat the justification of Step 4, as follows.

Each player $i \in N$ has no influence on $\boldsymbol{\theta}$ and $\boldsymbol{a}_{-i}$. Therefore, $f_i(\theta_i, \boldsymbol{a}_{-i})$ is independent of s_i, and (29) implies that i can get at least $f_i(\theta_i, \boldsymbol{a}_{-i})$ utility. Each player $i \in N$ has no hope of getting more expected utility than the maximum possible total utility of all players minus the sum of the guaranteed utilities of the other players. Therefore, if the players other than i take $\boldsymbol{a}_{-i}^*$, then i has no hope of getting more expected utility than

$$\sup_{\boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}} \sum_{j \in N} u_j(\boldsymbol{\theta}, \mathbf{s}) - \inf_{a_i \in \mathcal{A}_i} \sum_{j \in N-i} f_j\big(\theta_j, (a_i, \boldsymbol{a}_{-i-j}^*)\big)$$

$$\overset{(30)(31)}{=} \sum_{j \in N} f_j(\theta_j, \boldsymbol{a}_{-j}^*) - \sum_{j \in N-i} f_j(\theta_j, \boldsymbol{a}_{-j}^*) = f_i(\theta_i, \boldsymbol{a}_{-i}^*).$$

Consequently, each player $i \in N$ has no incentive to deviate from s_i^*, and therefore, we can rightfully say that $\boldsymbol{s}^*$ is an equilibrium. □

Step 6. There is a set of players N playing a stochastic dynamic game with complete information. At the very beginning, nature chooses a type $\theta_i \in \Theta_i$ for each player $i \in N$, as a nondeterministic action with no prior probabilities. Then at the same time point, the players do simultaneous actions, denoted by $a_i = a_i(\boldsymbol{\theta}, s_i) \in \mathcal{A}_i$ for each player $i \in N$. The further actions of nature are chosen with given probabilities. Suppose that there are functions $f_i : \Theta_i \times \boldsymbol{\mathcal{A}}_{-i} \to \mathbb{R}$ for all $i \in N$, and a strategy profile $\boldsymbol{s}^* \in \boldsymbol{\mathcal{S}tr}$ with $a_i^* = a_i(s_i^*)$, satisfying the following.

$$\forall \boldsymbol{\theta} \in \boldsymbol{\Theta},\ \forall i \in N,\ \forall \boldsymbol{s}_{-i} \in \boldsymbol{\mathcal{S}tr}_{-i}\colon \qquad \mathrm{E}\Big(u_i\big(\boldsymbol{\theta}, (s_i^*, \boldsymbol{s}_{-i})\big)\Big) \geq f_i\big(\theta_i, \boldsymbol{a}_{-i}(\boldsymbol{\theta}, \boldsymbol{s}_{-i})\big) \tag{32}$$

$$\forall \boldsymbol{\theta} \in \boldsymbol{\Theta},\ \forall \boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}\colon \qquad \sum_{i \in N} \mathrm{E}\big(u_i(\boldsymbol{\theta}, \mathbf{s})\big) \leq \sum_{i \in N} f_i(\theta_i, \boldsymbol{a}_{-i}^*) \tag{33}$$

$$\forall \boldsymbol{\theta} \in \boldsymbol{\Theta},\ \forall i \in N,\ \forall a_i \in \mathcal{A}_i\colon \qquad \sum_{j \in N-i} f_j\big(\theta_j, (a_i, \boldsymbol{a}_{-i-j}^*)\big) \geq \sum_{j \in N-i} f_j(\theta_j, \boldsymbol{a}_{-j}^*) \tag{34}$$

Then $\boldsymbol{s}^*$ is an equilibrium.

Justification. Each player $i \in N$ has no influence on $\boldsymbol{\theta}$ and $\boldsymbol{a}_{-i}$. Therefore, $f_i(\theta_i, \boldsymbol{a}_{-i})$ is independent of s_i, and (32) implies that i can get at least $f_i(\theta_i, \boldsymbol{a}_{-i})$ expected utility. Each player $i \in N$ has no hope of getting more expected utility than the maximum possible total expected utility of all players minus the sum of the guaranteed expected utilities of the other players. Therefore, if the players other than i take $\boldsymbol{a}^*_{-i}$, then i has no hope of getting more expected utility than

$$\sup_{\boldsymbol{s}\in\boldsymbol{\mathcal{S}tr}}\sum_{j\in N}\mathrm{E}\big(u_j(\boldsymbol{\theta},\mathrm{s})\big)-\inf_{a_i\in\mathcal{A}_i}\sum_{j\in N-i}f_j\big(\theta_j,(a_i,\boldsymbol{a}^*_{-i-j})\big)$$

$$\overset{(33)(34)}{=}\sum_{j\in N}f_j(\theta_j,\boldsymbol{a}^*_{-j})-\sum_{j\in N-i}f_j(\theta_j,\boldsymbol{a}^*_{-j})=f_i(\theta_i,\boldsymbol{a}^*_{-i}).$$

Consequently, each risk-neutral player $i\in N$ has no incentive to deviate from s^*_i, and therefore, we can rightfully say that $\boldsymbol{s}^*$ is an equilibrium. □

Step 7. Consider the same game as in Step 6, but with a strategy set $\mathcal{S}tr'_i$ for each $i\in N$ satisfying $s^*_i\in\mathcal{S}tr'_i\subseteq\mathcal{S}tr_i$. Then $\boldsymbol{s}^*$ is also an equilibrium in this game.

Justification. The same justification works here as for Step 6. □

If a strategy profile satisfies the conditions above, then we say that it is a **dynamic quasi-dominant Nash-equilibrium**.

We apply this equilibrium to our setup and the mechanisms. The first same-time actions of the players are identified with the applications $\boldsymbol{a}_{\mathcal{P}l}=\hat{\boldsymbol{\theta}}_{\mathcal{P}l}$, where the principal has no other choice than $\hat{\theta}_P=\theta_P$. Then a profile of simple strategies $\boldsymbol{s}^*$ with $a^*_i(\boldsymbol{\theta},\boldsymbol{s}^*)=\boldsymbol{\theta}$ is a dynamic quasi-dominant Nash-equilibrium if there exist functions $f_y:\Theta\to\mathbb{R}$ for all player $y\in\mathcal{P}l$, so that the followings hold.

$$\forall\boldsymbol{\theta}\in\boldsymbol{\Theta},\forall i\in\mathcal{A}g,\ \forall\boldsymbol{s}_{\mathcal{P}l-i}\in\boldsymbol{\mathcal{S}tr}_{\mathcal{P}l-i}:\quad \mathrm{E}\Big(u_i\big(\boldsymbol{\theta},(s^*_i,\boldsymbol{s}_{-i})\big)\Big)\ge f_i\Big(\big(\theta_i,\hat{\boldsymbol{\theta}}_{-i}(\boldsymbol{\theta},\boldsymbol{s}_{-i})\big)\Big) \tag{35}$$

$$\forall\boldsymbol{\theta}\in\boldsymbol{\Theta},\ \forall\boldsymbol{s}_{\mathcal{A}g}\in\boldsymbol{\mathcal{S}tr}_{\mathcal{A}g}:\quad \mathrm{E}\Big(u_P\big(\boldsymbol{\theta},(s^*_P,\boldsymbol{s}_{\mathcal{A}g})\big)\Big)\ge f_P\Big(\big(\theta_P,\hat{\boldsymbol{\theta}}_{\mathcal{A}g}(\boldsymbol{\theta},\boldsymbol{s}_{\mathcal{A}g})\big)\Big) \tag{36}$$

$$\forall\boldsymbol{\theta}\in\boldsymbol{\Theta},\ \forall\boldsymbol{s}\in\boldsymbol{\mathcal{S}tr}:\quad \mathrm{E}\big(u_{\mathcal{P}l}(\boldsymbol{\theta},\mathrm{s})\big)\le\sum_{y\in\mathcal{P}l}f_y(\boldsymbol{\theta}) \tag{37}$$

$$\forall\boldsymbol{\theta}\in\boldsymbol{\Theta},\ \forall i\in\mathcal{A}g,\ \forall\hat{\theta}_i\in\Theta_i:\quad \sum_{y\in\mathcal{P}l-i}f_y\big((\boldsymbol{\theta}_{-i},\hat{\theta}_i)\big)\ge\sum_{y\in\mathcal{P}l-i}f_y(\boldsymbol{\theta}) \tag{38}$$

5.3 The proof with dynamic quasi-dominant Nash-equilibrium

Theorem 9. *The cost price strategy profile $\boldsymbol{cp}=\boldsymbol{cp}_{\mathcal{A}g}$ is a dynamic quasi-dominant Nash-equilibrium under the second price mechanism.*

Proof. We show that the following functions f_y satisfy (35), (36), (37) and (38).

$$f_i(\theta_i,\hat{\boldsymbol{\theta}}_{-i})=V\big((\theta_i,\hat{\boldsymbol{\theta}}_{-i})\big)-V(\hat{\boldsymbol{\theta}}_{-i}) \tag{39}$$

$$f_P(\theta_P,\hat{\theta}_{\mathcal{A}g})=\sum_{i\in\mathcal{A}g}V(\hat{\boldsymbol{\theta}}_{-i})-(n-1)V(\hat{\boldsymbol{\theta}}) \tag{40}$$

Proof of (35). $\forall\boldsymbol{\theta}\in\boldsymbol{\Theta},\ \forall i\in\mathcal{A}g,\ \forall\boldsymbol{s}_{\mathcal{A}g-i}\in\boldsymbol{\mathcal{S}tr}_{\mathcal{A}g-i}$:

$$\mathrm{E}\Big(u_i^{2nd}\big(\boldsymbol{\theta},(cp_i,\boldsymbol{s}_{-i})\big)\Big)\overset{(10)}{=}\mathrm{E}\Big(u_i\big(\boldsymbol{\theta},(cp_i,\boldsymbol{s}_{-i})\big)+v_i^+(\hat{\boldsymbol{\theta}})\Big)=\mathrm{E}\Big(u_i\big(\boldsymbol{\theta},(cp_i,\boldsymbol{s}_{-i})\big)\Big)+v_i^+\big((\theta_i,\hat{\theta}_i)\big)$$

$$\overset{(16)}{=}v_i^+\big((\theta_i,\hat{\theta}_i)\big)\overset{(9)}{=}V\big((\theta_i,\hat{\boldsymbol{\theta}}_{-i})\big)-V(\hat{\boldsymbol{\theta}}_{-i})\overset{(39)}{=}f_i(\theta_i,\hat{\boldsymbol{\theta}}_{-i})$$

Proof of (36). $\forall \boldsymbol{\theta} \in \boldsymbol{\Theta},\ \forall \boldsymbol{s}_{\mathcal{A}g} \in \boldsymbol{\mathcal{S}tr}_{\mathcal{A}g}$:

$$\mathrm{E}\big(u_P^{2nd}(\boldsymbol{\theta}, \boldsymbol{s}_{\mathcal{A}g})\big) \overset{(11)}{=} \mathrm{E}\big(u_P(\boldsymbol{\theta}, \boldsymbol{s}_{\mathcal{A}g})\big) - \sum_{i\in\mathcal{A}g} v_i^+(\hat{\boldsymbol{\theta}}) \overset{(4)(9)}{=} V(\hat{\boldsymbol{\theta}}) - \sum_{i\in\mathcal{A}g} \big(V(\hat{\boldsymbol{\theta}}) - V(\hat{\boldsymbol{\theta}}_{-i})\big)$$

$$= \sum_{i\in\mathcal{A}g} V(\hat{\boldsymbol{\theta}}_{-i}) - (n-1)V(\hat{\boldsymbol{\theta}}) \overset{(40)}{=} f_P(\theta_P, \hat{\boldsymbol{\theta}}_{\mathcal{A}g})$$

Proof of (37). $\mathcal{A}_P = \{a_P^*\}$, therefore, (37) holds for the principal. Furthermore, $\forall \boldsymbol{\theta} \in \boldsymbol{\Theta},\ \forall \boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}$:

$$\mathrm{E}\big(u_{\mathcal{P}l}(\boldsymbol{\theta}, \mathbf{s})\big) \overset{(4)}{\leq} V(\boldsymbol{\theta}) = \sum_{i\in\mathcal{A}g} \big(V(\boldsymbol{\theta}) - V(\boldsymbol{\theta}_{-i})\big) + \Big(\sum_{i\in\mathcal{A}g} V(\boldsymbol{\theta}_{-i}) - (n-1)V(\boldsymbol{\theta})\Big)$$

$$\overset{(39)(40)}{=} \sum_{i\in\mathcal{A}g} f_i(\theta_i, \boldsymbol{\theta}_{-i}) + f_P(\theta_P, \boldsymbol{\theta}_{\mathcal{A}g}) = \sum_{y\in\mathcal{P}l} f_y\big(\theta_y, \hat{\boldsymbol{\theta}}_{-y}(\boldsymbol{\theta}, \boldsymbol{cp})\big)$$

Proof of (38). $\forall \boldsymbol{\theta} \in \boldsymbol{\Theta},\ \forall i \in \mathcal{A}g,\ \forall \hat{\theta}_i \in \Theta_i$:

$$\sum_{y\in\mathcal{P}l-i} f_y\big(\theta_y, (\hat{\theta}_i, \boldsymbol{\theta}_{\mathcal{A}g-i-y})\big) = f_P\big(\theta_y, (\hat{\theta}_i, \boldsymbol{\theta}_{\mathcal{A}g-i})\big) + \sum_{j\in\mathcal{A}g-i} f_j\big(\theta_j, (\hat{\theta}_i, \boldsymbol{\theta}_{\mathcal{A}g-i-j})\big) \overset{(40)(39)}{=}$$

$$\sum_{j\in\mathcal{A}g-i} V\big((\hat{\theta}_i, \boldsymbol{\theta}_{-i-j})\big) + V(\boldsymbol{\theta}_{-i}) - (n-1)V\big((\hat{\theta}_i, \boldsymbol{\theta}_{-i})\big) + \sum_{j\in\mathcal{A}g-i} V\big((\hat{\theta}_i, \boldsymbol{\theta}_{-i})\big) - V\big((\hat{\theta}_i, \boldsymbol{\theta}_{-i-j})\big) = V(\boldsymbol{\theta}_{-i}),$$

therefore, (38) holds with equation. □

6 Efficiency and incentives under the first price mechanism

The message is the following. The relation between the first and second price mechanisms in our model is similar to the relation between first and second price sealed bid single item or combinatorial auction.

It is known that while second price auctions implement the social optimum with dominant strategies, we have no that clear mathematical results for first price auctions. However, in practice, it turns out that first price auctions are close to being efficient, and these have better properties from other aspects, for example, these often provide higher revenue to the seller. Therefore, first price auctions are more general in practice than second price auctions.

In this section, we try to justify the "almost efficiency" of the first price mechanism. We note that the first price sealed bid combinatorial auction (FPCA) is a special case of the first price mechanism in our model, therefore, we cannot expect stronger properties than those of the FPCA.

We say that there is **perfect competition** if it is publicly known that the type vector $\boldsymbol{\theta}$ is chosen from

$$\boldsymbol{\Theta}^{perf} = \big\{\boldsymbol{\theta} \in \boldsymbol{\Theta} \bigm| \forall i \in \mathcal{A}g : V(\boldsymbol{\theta}_{-i}) = V(\boldsymbol{\theta})\big\} \tag{41}$$

Theorem 10. *The cost price strategy profile $\boldsymbol{cp} = \boldsymbol{cp}_{\mathcal{A}g}$ under the first price mechanism is a dynamic quasi-dominant Nash-equilibrium under perfect competition.*

Proof. We show that the following functions f_y satisfy (35), (36), (37) and (38) with "$\forall\boldsymbol{\theta} \in \boldsymbol{\Theta}^{perf}$" instead of "$\forall\boldsymbol{\theta} \in \boldsymbol{\Theta}$".

$$f_i(\theta_i, \hat{\boldsymbol{\theta}}_{\mathcal{A}g-i}) = 0 \tag{42}$$

$$f_P(\theta_P, \hat{\boldsymbol{\theta}}_{\mathcal{A}g}) = V\big((\theta_P, \hat{\boldsymbol{\theta}}_{\mathcal{A}g})\big) = V(\hat{\boldsymbol{\theta}}) \tag{43}$$

Proof of (35). $\forall\boldsymbol{\theta} \in \boldsymbol{\Theta}^{perf},\ \forall i \in \mathcal{A}g,\ \forall \boldsymbol{s}_{\mathcal{A}g-i} \in \boldsymbol{\mathcal{S}tr}_{\mathcal{A}g-i}:$

$$\mathrm{E}\Big(u_i\big(\boldsymbol{\theta}, (cp_i, \boldsymbol{s}_{-i})\big)\Big) \overset{(16)}{=} 0 \overset{(42)}{=} f_i(\theta_i, \hat{\boldsymbol{\theta}}_{\mathcal{A}g-i})$$

Proof of (36). $\forall\boldsymbol{\theta} \in \boldsymbol{\Theta}^{perf},\ \forall \boldsymbol{s}_{\mathcal{A}g} \in \boldsymbol{\mathcal{S}tr}_{\mathcal{A}g}:$

$$\mathrm{E}\big(u_P(\boldsymbol{\theta}, \boldsymbol{s}_{\mathcal{A}g})\big) \overset{(4)}{=} V(\hat{\boldsymbol{\theta}}) \overset{(43)}{=} f_P(\theta_P, \hat{\boldsymbol{\theta}}_{\mathcal{A}g})$$

Proof of (37). $\forall\boldsymbol{\theta} \in \boldsymbol{\Theta}^{perf},\ \forall \boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}:$

$$\mathrm{E}\big(u_{\mathcal{P}l}(\boldsymbol{\theta}, \mathrm{s})\big) \overset{(4)}{\leq} V(\boldsymbol{\theta}) = V\big(\hat{\boldsymbol{\theta}}(\boldsymbol{\theta}, \boldsymbol{cp})\big) \overset{(43)}{=} f_P\big(\theta_P, \hat{\boldsymbol{\theta}}_{\mathcal{A}g}(\boldsymbol{\theta}, \boldsymbol{cp})\big) \overset{(42)}{=} \sum_{y \in \mathcal{P}l} f_y\big(\theta_y, \hat{\boldsymbol{\theta}}_{-y}(\boldsymbol{\theta}, \boldsymbol{cp})\big)$$

Proof of (38). $\forall\boldsymbol{\theta} \in \boldsymbol{\Theta}^{perf},\ \forall i \in \mathcal{A}g,\ \forall \hat{\theta}_i \in \Theta_i:$

$$\sum_{y \in \mathcal{P}l-i} f_y\big(\theta_y, (\hat{\theta}_i, \boldsymbol{\theta}_{\mathcal{A}g-i-y})\big) \overset{(42)}{=} f_P\big(\theta_P, (\hat{\theta}_i, \boldsymbol{\theta}_{\mathcal{A}g-i})\big) \overset{(43)}{=} V\big((\hat{\theta}_i, \boldsymbol{\theta}_{-i})\big)$$

$$\geq V(\boldsymbol{\theta}_{-i}) \overset{(41)}{=} V(\boldsymbol{\theta}) \overset{(43)}{=} f_P(\theta_P, \boldsymbol{\theta}_{\mathcal{A}g}) \overset{(42)}{=} \sum_{y \in \mathcal{P}l-i} f_y\big(\theta_y, (\boldsymbol{\theta}_{\mathcal{A}g-y})\big),$$

where the step $V\big((\hat{\theta}_i, \boldsymbol{\theta}_{-i})\big) \geq V(\boldsymbol{\theta}_{-i})$ can formally be derived from (45). □

For any agent i, let G_i, G_{-i} and G_S denote the game G with the restriction that the principal must accept i, must not accept i, or must accept all agents in S and no one else, respectively. The first price mechanism in each of these games is defined in the same way as in G, namely, we only change G to the corresponding game in (6). Let $M_i = G_i/m_1$ and $M_{-i} = G_{-i}/m_1$ and $M_S = G_S/m_1$.

Notice that the proofs of Lemmas 4 and 5 are valid in these games, as well. Namely, for any $M_* \in \{M_i, M_{-i}, M_S\}$,

$$u^{M_*}(P) = V\big(M_*(\hat{\boldsymbol{\theta}})\big), \tag{44}$$

$$\mathrm{E}\big(u_i^{M_*}(cp_i)\big) = 0.$$

Notice that

$$V(G) \overset{(4)}{=} \sup_{\boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}} \mathrm{E}\big(u_{\mathcal{P}l}(\mathrm{s})\big) = \max\Big(\sup_{\boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}\ |\ i \in \mathcal{A}cc} \mathrm{E}\big(u_{\mathcal{P}l}(\mathrm{s})\big), \sup_{\boldsymbol{s} \in \boldsymbol{\mathcal{S}tr}\ |\ i \notin \mathcal{A}cc} \mathrm{E}\big(u_{\mathcal{P}l}(\mathrm{s})\big)\Big)$$

$$\overset{(4)}{=} \max\big(V(G_i), V(G_{-i})\big),$$

and the principal accepts i if and only if $V\big(G_i(\hat{\boldsymbol{\theta}})\big) > V\big(G_{-i}(\hat{\boldsymbol{\theta}})\big)$. Notice that $V\big(G_{-i}(\hat{\boldsymbol{\theta}})\big) = V(\hat{\boldsymbol{\theta}}_{-i})$, because it does not matter whether i does not play at all or whether he takes part in the game but the principal rejects him for sure. Therefore,

$$V(\hat{\boldsymbol{\theta}}) = \max\Big(V\big(G_i(\hat{\boldsymbol{\theta}})\big), V(\hat{\boldsymbol{\theta}}_{-i})\Big). \tag{45}$$

Let the **signed value of an application** $\hat{\theta}_i$ be

$$v_i^{\pm} = v_i^{\pm}(\hat{\boldsymbol{\theta}}) = v_i^{\pm}(\hat{\theta}_i) = V\big(G_i(\hat{\boldsymbol{\theta}})\big) - V\big(G_{-i}(\hat{\boldsymbol{\theta}})\big) = V\big(G_i(\hat{\boldsymbol{\theta}})\big) - V(\hat{\boldsymbol{\theta}}_{-i}). \tag{46}$$

Then

$$v_i^{+} \overset{(9)}{=} V(\hat{\boldsymbol{\theta}}) - V(\hat{\boldsymbol{\theta}}_{-i}) \overset{(45)}{=} \max\Big(V(\hat{\boldsymbol{\theta}}_{-i}), V\big(G_i(\hat{\boldsymbol{\theta}})\big)\Big) - V(\hat{\boldsymbol{\theta}}_{-i}) \overset{(46)}{=} \max(0, v_i^{\pm}), \tag{47}$$

and $v_i^{\pm} > 0$ if and only if $i \in \mathcal{A}cc$.

For a type $\theta_i = (d_i, v_i)$ and a number x which depends on $\boldsymbol{\theta}$ (the information of the agent when he submits his application), let $\theta_i - x = (d_i, v_i - x)$. In addition, for a strategy s_i using application $\hat{\theta}_i$, let $s_i - x$ mean the strategy by which i always makes the same actions as he would do with s_i, except that he sends the application $\hat{\theta}_i - x$ instead. We call a strategy $cp_i - x$ the **fair strategy** and an application $\theta_i - x$ a **fair application** with **profit** x. Notice that if $s_i = cp_i - x$, then

$$\hat{v}_i(\hat{l}_i, \boldsymbol{c}) = v_i(l_i, \boldsymbol{c}) - x \tag{48}$$

Let a **fair-like strategy** of an agent $i \in \mathcal{A}g$ mean a strategy by which i sends an application $\hat{\theta}_i$, and then he behaves as follows. The chance event he reports has the same probability distribution as in $\hat{\theta}_i$ which is independent of the preceding actions of all other players, and in the end, he achieves the consequence corresponding to the communication. In other words, he behaves as an agent with type $\hat{\theta}_i$ who uses cost price strategy.

Clearly, the cost price strategy is the fair strategy with profit 0, and all fair strategies are fair-like strategies. The primary example of fair-like strategies is when the agent applies with his real type but with valuation decreased by different amounts at different executions, and then he plays truthfully and obediently.

Similarly to Lemma 5, the following lemma shows that an agent with fair strategy has guaranteed expected utility in the case of acceptance.

Lemma 11. *If $i \in \mathcal{A}cc$, then*

$$\mathrm{E}\big(u_i(s_i = cp_i - x)\big) = x \tag{49}$$

Proof.

$$\mathrm{E}\big(u_i\big) \overset{(1)}{=} \mathrm{E}\Big(v_i(l_i, \boldsymbol{c}) + p_i\Big) \overset{(8)}{=} \mathrm{E}\Big(v_i(l_i, \boldsymbol{c}) - \hat{v}_i(\hat{l}_i, \boldsymbol{c}) + \sum_{\chi \in X_i} \delta(\chi)\Big) \overset{(14)}{=} \mathrm{E}\Big(v_i(l_i, \boldsymbol{c}) - \hat{v}_i(\hat{l}_i, \boldsymbol{c})\Big) \overset{(48)}{=} x \quad \square$$

The following theorem shows that if all agents use fair strategy, then after the choice of $\mathcal{A}cc$, the mechanism works efficiently. This implies that the principal chooses the set $\mathcal{A}cc$ for which the expected social welfare *minus the sum of the demanded profits of the agents in* $\mathcal{A}cc$ is the largest possible; and this is the only inefficient decision throughout the game.

Theorem 12. *If $\forall i \in \mathcal{A}cc$: $s_i = cp_i - x_i$, then*

$$\mathrm{E}(u_{\mathcal{P}l}) = V(G_{\mathcal{A}cc}) \tag{50}$$

Proof 1. Notice that the only difference between $G_{\mathcal{A}cc}(\boldsymbol{\theta}) \cong G_{\mathcal{A}cc}(\boldsymbol{\theta}_{\mathcal{A}cc})$ and $G_{\mathcal{A}cc}(\hat{\boldsymbol{\theta}}) \cong G_{\mathcal{A}cc}(\hat{\boldsymbol{\theta}}_{\mathcal{A}cc})$ is that each agent $i \in \mathcal{A}cc$ gets x_i higher utility in the latter game. Hence,

$$u^{M_{\mathcal{A}cc}}(P) \overset{(44)}{=} V\big(G_{\mathcal{A}cc}(\hat{\boldsymbol{\theta}}_{\mathcal{A}cc})\big) = V(G_{\mathcal{A}cc}) - \sum_{i \in \mathcal{A}cc} x_i, \tag{51}$$

therefore, in $M_{\mathcal{A}cc}$,

$$\mathrm{E}(u_{\mathcal{P}l}) = \mathrm{E}(u_P) + \sum_{i \in \mathcal{A}cc} \mathrm{E}(u_i) \overset{(51)(49)}{=} \Big(V(G_{\mathcal{A}cc}) - \sum_{i \in \mathcal{A}cc} x_i\Big) + \sum_{i \in \mathcal{A}cc} x_i = V(G_{\mathcal{A}cc}). \quad \square$$

Proof 2. The reported execution maximizes the reported $\mathrm{E}\big(u^{M_{\mathcal{A}cc}}(\mathcal{P}l)\big)$ in the reported game $M_{\mathcal{A}cc}(\hat{\boldsymbol{\theta}})$. If $\forall i \in \mathcal{A}cc$: $s_i = cp_i - x_i$, then the only difference between $M_{\mathcal{A}cc}(\boldsymbol{\theta})$ and $M_{\mathcal{A}cc}(\hat{\boldsymbol{\theta}})$ is that each agent $i \in \mathcal{A}g$ gets x_i higher utility in the latter case, and this remains the only difference between the actual and the reported states of the project throughout the game. Consequently,

$$\mathrm{E}\Big(u_{\mathcal{P}l}(\forall i \in \mathcal{A}cc\colon s_i = cp_i - x_i)\Big) = \mathrm{E}\Big(u_{\mathcal{P}l}^{M(\boldsymbol{\theta}_{\mathcal{A}cc})}\big((cp_i - x_i)_{i\in\mathcal{A}cc}\big)\Big)$$

$$= \mathrm{E}\Big(u_{\mathcal{P}l}^{M(\hat{\boldsymbol{\theta}}_{\mathcal{A}cc})}\big(\boldsymbol{cp}(\hat{\boldsymbol{\theta}}_{\mathcal{A}cc})\big)\Big) + \sum_{i\in\mathcal{A}cc} x_i \overset{(17)}{=} V\big(G(\hat{\boldsymbol{\theta}}_{\mathcal{A}cc})\big) + \sum_{i\in\mathcal{A}cc} x_i = V\big(G(\boldsymbol{\theta}_{\mathcal{A}cc})\big). \qquad \square$$

$V\Big(G_i\big((\hat{\theta}_i - x, \hat{\boldsymbol{\theta}}_{-i})\big)\Big) = V\big(G_i(\hat{\boldsymbol{\theta}})\big) - x$ because $G_i\big((\hat{\theta}_i - x, \hat{\boldsymbol{\theta}}_{-i})\big)$ is the same game as $G_i(\hat{\boldsymbol{\theta}})$ but in the former game, agent i gets x less utility at the end. Thus,

$$v_i^{\pm}(\hat{\theta}_i - x) \overset{(46)}{=} V\Big(G_i\big((\hat{\theta}_i - x, \hat{\boldsymbol{\theta}}_{-i})\big)\Big) - V(\hat{\boldsymbol{\theta}}_{-i}) = V\big(G_i(\hat{\boldsymbol{\theta}})\big) - x - V(\hat{\boldsymbol{\theta}}_{-i}) \overset{(46)}{=} v_i^{\pm}(\hat{\theta}_i) - x.$$

Thus, for a particular agent, there is exactly one fair application with a given signed value.

Let the **strategy form** of a strategy s_i mean $\varphi(s_i) = \{s_i + x \mid x \in \mathbb{R}\}$. Let $\varphi(cp_i) = \{cp_i + x \mid x \in \mathbb{R}\}$ called the **fair strategy form**.

Given an agent i and strategy form φ, provided that $i \in \mathcal{A}cc$, which strategy i uses from φ has no direct effect on the other agents. Namely, they face the very same subgame after sending the applications, except a constant difference in the payment from P to i at the end. Therefore, we assume that if an agent i uses another strategy from the same strategy form, and his application is accepted in both cases, then the execution of the game remains the same beyond the constant difference in the payment. This assumption automatically holds if the applications are not observable for the other agents.

Given $\boldsymbol{\theta}$ and $\boldsymbol{s}_{-i}$, we have that $\mathrm{E}\big(u_i^{M_i}(s_i + x)\big) = \mathrm{E}\big(u_i^{M_i}(s_i)\big) - x$. Furthermore, $v_i^{\pm}(\hat{\theta}_i + x) = v_i^{\pm}(\hat{\theta}_i) + x$. These imply that $\mathrm{E}\big(u_i^{M_i}(s_i)\big) + v_i^{\pm}(\hat{\theta}_i)$ depends only on $\varphi(s_i)$. We call this sum the **value of the strategy form** $V_i\big(\varphi(s_i)\big)$.

In each strategy form, there exists a strategy s_i, using an application $\hat{\theta}_i$, for which $\hat{\theta}_i + x$ would be accepted if $x > 0$ and rejected if $x < 0$. Then

$$\mathrm{E}\big(u_i(s_i + x)\big) = \{0 \text{ if } x < 0;\ \text{and } V_i\big(\varphi(s_i)\big) - x \text{ if } x > 0\}.$$

Theorem 13. *Consider an arbitrary agent $i \in \mathcal{A}g$. Suppose that every other agent $j \in \mathcal{A}cc - i$ uses fair-like strategy s_j. Then the fair strategies of i maximize the value of the strategy form*

$$v_i^{\pm}\Big(\hat{\theta}\big((s_i, \boldsymbol{s}_{-i})\big)\Big) + \mathrm{E}\Big(u_i^{M_i}\big((s_i, \boldsymbol{s}_{-i})\big)\Big).$$

Proof. Notice that whether an agent j has a type Θ_j and he uses a fair-like strategy s_j, or his type is $\Theta_j(\hat{\Theta}_j, s_j)$ and he uses cost price strategy, these are the same from all the other players' point of view. Therefore, from the point of view of i, the only difference between $M(\boldsymbol{\theta})$ with fair-like strategies $\boldsymbol{s}_{-i}$ and $M' = M\big((\theta_i + x, \hat{\boldsymbol{\theta}}_{-i}(\boldsymbol{s}_{-i}))\big)$ with cost price strategies $\boldsymbol{s}_{-i}^{M'} = \boldsymbol{cp}_{-i}^{M'}$ is that the utility of i is lower by x and the signed value of his application is higher by x in the latter case. Thus, after this transformation of the game M to M' with an x large enough, we only need to prove the following. If $\boldsymbol{s}_{-i} = \boldsymbol{cp}_{-i}$ and any s_i with application $\hat{\theta}_i$ satisfies $v_i^{\pm}(\boldsymbol{\theta}) = v_i^{\pm}\big((\hat{\theta}_i, \boldsymbol{\theta}_{-i})\big) > 0$, then $\mathrm{E}\big(u_i^{M_i}(s_i, \boldsymbol{cp}_{-i})\big) \le \mathrm{E}\big(u_i^{M_i}(\boldsymbol{cp})\big)$.

For a large x, $i \in \mathcal{A}cc$, therefore,

$$\mathrm{E}\Big(u_i^{M_i}\big((s_i, \boldsymbol{cp}_{-i})\big)\Big) = \mathrm{E}\Big(u_i\big((s_i, \boldsymbol{cp}_{-i})\big)\Big)$$

$$= \mathrm{E}\Big(u_{\mathcal{P}l}\big((s_i, \boldsymbol{cp}_{-i})\big)\Big) - \mathrm{E}\Big(u_P\big((s_i, \boldsymbol{cp}_{-i})\big)\Big) - \sum_{j \in \mathcal{A}g - i} \mathrm{E}\Big(u_j\big((s_i, \boldsymbol{cp}_{-i})\big)\Big)$$

$$\overset{(5)(15)(16)}{\leq} V(G) - V\Big(\hat{\theta}\big((s_i, \boldsymbol{cp}_{-i})\big)\Big) = V(G) - V\Big(G_i\big(\hat{\boldsymbol{\theta}}((s_i, \boldsymbol{cp}_{-i}))\big)\Big) \overset{(46)}{=} V(G) - V(\hat{\boldsymbol{\theta}}_{-i}) - v_i^{\pm}$$

and Theorem 6 shows that equality holds if $s_i = cp_i$. □

$\mathrm{E}\big(u_i(s_i)\big) \leq V_i\big(\varphi(s_i)\big)$, therefore, the strategy form of $i \in \mathcal{A}g$ with the highest value gains him the highest potential for his expected utility. Theorem 13 implies that the fair strategy form has the highest value. Moreover,

$$\mathrm{E}\big(u_i(cp_i - x)\big) = \big\{x \text{ if } V_i(cp_i) > x; \text{ and } 0 \text{ otherwise}\big\},$$

which is quite an efficient way of the exploitation of this potential. Of course, this only shows that the fair strategy with appropriate profit is approximately the best for an agent, under competition.

In Section A.4, we show one more argument about why we expect that the agents would use approximately fair applications in real-life situations.

6.1 Intuitive interpretation

We tried to show the most convincing formal reasons why the first price mechanism is close to efficient. Now we present an intuitive summary.

Consider a case when an agent $i \in \mathcal{A}g$ is asked to work on either a larger or a smaller task, and this choice depends mainly on the applications of others. Then i should demand more profit (payment beyond his costs) in the former case and less profit in the latter case, due to the different competition settings in the two cases. Similar phenomena may occur in other cases, as well.

The opinion of the author is that a rational agent would use a strategy very similar to a fair strategy, mainly with deviations like in the previous case. This behavior would still provide approximate efficiency.

We describe a situation when an agent should use a highly non-fair-like strategy, but it requires a very extreme information structure, and does not cause big inefficiency. Consider an agent A applying for a subtask. Suppose that agent A knows that his type is clearly the best one for the task, and agent B is the clearly the second best. Suppose that A knows very well the type and information of B, but B believes (with probability almost 1) that it is not the case. Therefore, A can predict well the application of B. If A does not know the types of agents applying for other tasks, and believes that they do not know about this special situation, then A should apply with the application of agent B but claiming slightly smaller payment, in order to very slightly underbid B. He should combine it with the techniques described in Appendices B.1, B.2, B.3, which resolve all problems, and with this strategy (and with no other strategy), A can get almost the maximum possible expected utility $V\big(\theta_A, \hat{\boldsymbol{\theta}}_{-A})\big) - V(\hat{\boldsymbol{\theta}}_{-A})$.

Notice that the information structure of the example above is exceptional in a sense that under common prior assumption, this must have very low probability. We do not even know an unrealistic example of a common prior when first price mechanism can be very inefficient. In more detail, see Appendix A.4.

7 Advantages of the first price mechanism over the second price mechanism

7.1 Collusion

In this section, we consider the case when we have disjoint sets of agents, called coalitions, and each agent in a coalition knows (at most) their total information and prefers the higher total expected utility of all agents in his coalition.

The first price mechanism. There is a natural definition of product of types (described in Examples 1.1, 1.2). The execution of its workflow corresponds to the parallel executions of the original workflows, and the valuation is the sum of the original valuations. Correspondingly, for an arbitrary set of agents, denoted by $\{1, 2, ..., k\}$, product $\boldsymbol{s}_\pi$ of strategies $s_1, s_2, ..., s_k$ means the following strategy of a new agent π with the corresponding product of types $\boldsymbol{\theta}_\pi = \times_{i=1}^k \theta_i$. By this strategy, π makes all actions that each i would do with type θ_i and strategy s_i.

Consider now a coalition $\{1, 2, ..., k\}$. Consider the case when this coalition plays as one agent π with type $\boldsymbol{\theta}_\pi$ and strategy $\boldsymbol{s}_\pi$. It is easy to see that $\times_{i=1}^k \mathcal{S}tr_i \subset \boldsymbol{\mathcal{S}tr}_\pi$, that is, they can simulate the case when they play as different agents in a coalition. As a consequence, if the competition remains perfect with this new set of players, then the mechanism remains efficient.

The second price mechanism. Consider a case when two agents can send applications written in such a way that their individual work is useless without each other. For example, one agent just have a pink dummy spaceship in neo-Hawaiian style, and he offers to bring it; and the other agent offers to make the task provided that he gets a pink dummy spaceship in neo-Havaiian style, as a tool for his work. Assume that these applications are accepted. If either of them increased his reported valuation (decreased his reported costs) by x, then the value of both applications would increase by x. Consequently, they would get $2x$ more second price compensation in total, which means x more total utility for them. Using this trick, these players can get as much payment as they want.

7.2 Reliance on the principal

The mechanisms can be modified as follows. The players must use the communication protocol defined by the mechanism, but right before each chance node of each agent, the principal sends $\delta(\chi)$ for each possible chance event χ, which must satisfy $\sum w(\chi)\delta(\chi) = 0$. Choosing the same $\delta(\chi) = V(T_+^\chi - T^\chi)$ provides an equivalent execution of the game, with the only difference that $\delta(\chi)$ becomes a part of the basic information of the corresponding agent. But since he could calculate all these amounts from his information, his information remains equivalent. Therefore, our results remain valid in this modified model, as well.

The first price mechanism. This modification makes p_i a function of c_i and the communication between i and P. Now assume that the payments are determined by the mechanism, but the principal is free to choose all her other actions. Namely, she is free to choose $\mathcal{A}cc$, she is free to choose which moves to ask these agents to make, she can make arbitrary moves in her own workflow, and she can also choose the functions $\delta(\chi)$ satisfying $\sum w(\chi)\delta(\chi) = 0$.

Let cp_S denote the strategy of the principal by which he does the same as in the original model. We show two theorems which shows that the principal is not interested in any deviation.

Theorem 14. *In the extended model, under the first price mechanism and under perfect competition, the cost price strategy profile is a dynamic quasi-dominant Nash-equilibrium.*

Proof. Notice that (16) remains true with the same proof. Then the proof of Theorem 9 becomes valid in this case, but we need to prove (38) with $i = P$, namely,

$$\forall \boldsymbol{\theta} \in \boldsymbol{\Theta},\ \forall \hat{\theta}_P \in \Theta_P: \qquad \sum_{i\in\mathcal{A}g} f_i\big((\boldsymbol{\theta}_{\mathcal{A}g}, \hat{\theta}_P)\big) \ge \sum_{i\in\mathcal{A}g} f_i(\boldsymbol{\theta}), \tag{52}$$

which is obvious, because the left hand side is nonnegative, and the right hand side is 0. □

The following theorem applies only to static equilibria. In our case, this is still meaningful, but we do not go into details.

Theorem 15. *In the modified model, if the principal knows that all agents use fair-like strategy, then the principal is not interested in deviating from her declared strategy.*

Proof. With the corresponding modifications to the types of the agents, the principal can consider the agents to use cost price strategies. Notice that (16) remains true with the same proof, even if the principal can make such deviations. Thus,

$$\mathrm{E}(u_P) = \mathrm{E}(u_{\mathcal{P}l}) - \sum_{i\in\mathcal{A}g} \mathrm{E}(u_i) \overset{(16)}{=} \mathrm{E}(u_{\mathcal{P}l}) \overset{(5)}{\le} V(G),$$

and Theorem 6 shows that equality holds if the principal uses her declared strategy. □

Also, notice that (13) easily follows that if an agent colludes with the principal, then it means that the agent should choose cost price strategy. This result combined with Section 7.1 implies that the same is true if multiple agents collude with the principal.

The second price mechanism. In theory (and more in theory than in practice), (38) may not hold, therefore, we need to assume that the principal does not lie about her type. Also, we need to assume that she does not collude with any agent. The applications should be public, or at least, we should solve the problem of verifying the true second price compensations. There is also one more technical problem: if the principal could find only an approximately efficient strategy profile, then it could be problematic determining the second price compensations.

8 Conclusion

We introduced a general model for multi-agent projects when each agent has a separate hidden working process with hidden actions and private chance events, but they have influence on each other through publicly observable actions and events. Despite the fact that the project requires a high-level of cooperation and the agents can cheat in various ways without the risk of being caught, we have designed a mechanism which incentivizes the players including the owner of the project in truthful behavior, in a very strong sense. We introduced two very similar mechanisms, and there is a tradeoff between them: the first price mechanism is better in almost all aspects which are important in practice, including collusion-resistance, but this is only approximately efficient. This is more a theoretical than a practical deficiency. The second price mechanism implements social efficiency in a very strong sense, but even two colluding players can make it very bad in practice. More formally, the results are the following.

Under the second price mechanism, the cost price strategy profile is a dynamic quasi-dominant Nash-equilibrium, and this way the expected social welfare is the largest possible.

Under the first price mechanism, with some assumption (or approximation), there is an equilibrium consisting of the same strategies but demanding money increased by a constant (their expected utilities in the case of acceptance) in the applications.

Both mechanisms are individually rational, and useless agents get utility 0.

The *disadvantage* of the first price mechanism over the second price one is that it requires competition on each task, and this is completely efficient only with perfect competition.

The *advantages* of the first price mechanisms are

- collusion does not decrease efficiency as long as it does not decrease the competition, while under the second price mechanism, if two agents can send applications that are useless without each other, then they may get as much utility as they want;
- the agents need (almost) no reliance on the principal, even if they never get to know the applications of the others.

Furthermore, there is a huge literature about why Vickrey's second price auction (1961) [15], and its generalizations [7] are worse than a first price auction in most practical situations (e.g. Rothkopf, Teisberg, Kahn, 1990 [14]), and these reasons may be valid here, too.

For numerous technical details showing how to use the mechanism in practice, see Appendices A and B.

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Appendix

A Further theoretical observations and special cases

A.1 An alternative approach to the mechanisms

As an alternative approach (which was the original approach of the author), we define the first price mechanism in the following way. We define the contract as a function that determines the payment between two players depending on the consequences of all players and the communication between the two players. We define an application to be an offer to the principal for contract. At the beginning, each agent sends an application to the principal, and then the principal chooses which offers to accept. After she receives the applications, she uses the strategy out of all her possible strategies by which *her minimum possible expected utility is the largest*, where expectation is *with respect only to her own chance events*. We call this her **maximin utility**. So she is absolutely mistrustful, that is, she always expects the worst possible joint behavior of all agents. Correspondingly, we define the joint value of a set of applications as this maximin utility of the principal.

The alternative second price mechanism is the same as the first price mechanism, but the principal pays v_i^+ more money to each agent i beyond the payment according to their contract.

With this approach, fair-like application means the offer for using the same communication protocol as the protocol defined by an original fair-like application under the original mechanism, but the principal sends the assignments δ^* right before each chance node, as in Section 7.2, and each agent i demands $-\hat{v}_i(\hat{l}_i, \boldsymbol{c}) + \sum_{\chi \in X_i} \delta^*(\chi)$.

We do not repeat the proofs of all the results with this approach, but we show some steps that illustrate how this mistrustful strategy makes sense.

For a contract made from a fair-like application, we can define the subcontract from a reported state of the project as the contract corresponding to the subworkflow (subtree) according to the current state of the project. If all contracts are derived from fair-like applications, then we can define the **value $V(T)$ of a reported state of the project** T at a time point after 0 as the principal's maximin utility with the subcontracts from the reported state of the project (and with only these contracts). For an arbitrary $S \subset \mathcal{A}g$, we show that $V(G_S)$ can be calculated here in the same way as with the original definition with the original approach.

The value of the reported starting state (just after 0) is the principal's maximin utility provided that she accepts some agents with the given set of applications. We can determine the value of all reported states of the project by recursion, using the following.

- The value of an end-state is the sum of the valuations at the executions.
- The value of the state of the project changes only at a node; values of states of the project between two neighboring nodes are the same.
- If the values after a decision node are given, then the value at this node is their maximum, because the principal will choose the more favorable option.
- **Lemma 16.** *If the reported state of the project at a chance event is T, the reported states of the project after this chance event are $T_1, T_2, ..., T_k$ and the reported probabilities are $w_1, w_2, ..., w_k$, respectively, then*

$$V(T) = \sum_{i=1}^{k} w_i \cdot V(T_i).$$

Proof. Let us consider the reported subgame T, and let $x = \sum w_i \cdot V(T_i)$. For calculation, assume that the reported chance event will be chosen randomly with the reported probabilities w_i. Furthermore, assume that the principal surely gets the value of the reported state of the project after the chance event; it does not affect the value of any of these reported states of the project. In this case, whichever assignment she chooses, her expected utility will be $\sum w_i \cdot V(T_i) + \sum w_i \cdot \delta^*(i) = x + 0 = x$, which implies $V(T) \leq x$. On the other hand, $\sum w_i \big(V(T_i) - x\big) = 0$, therefore, if the principal assigns $V(T_i) - x$ to the ith branch, then she gets x in all cases. □

Our recursion also shows that the principal would get the value of the reported starting state as a fixed utility.

From now on, we use this alternative approach. We note that each application in the original model can be matched with an equivalent application in this model, but this is not true in the other direction.

A.2 Simplifications in special cases

The messages the principal sends depend only on her not strictly earlier chance events and the previous messages she received. Thus, if an agent $i \in \mathcal{A}g$ is sure that the principal receives no message from anyone else and has no chance event in the time interval $I = [a, b]$, then, without decreasing the value of his application, i can ask the principal (in the application) not to send messages during I, but to already send him at a what these messages would be depending on the messages she would have received from i. Similarly, if i is sure that the principal will not send any messages to anyone else during I, then, without changing the value of his application, i can offer the following. i sends all his messages only at b, that he would have sent during I; meanwhile, instead of each message the principal would have sent to i during I, she should send him what message she would have sent originally, depending on what she would have received from i before.

As a consequence, consider a project consisting of two tasks, where the second task can only be started after the first one has been accomplished. We put this into our model in the following way. The consequence of each agent for the first task (called first agent) consists of his completion time C_1. The consequence of each second agent consists of his starting time S_2 and the time C_2 he completes; and his workflow starts with doing nothing until an optional time point S_2, and then he can start his work. The valuation function of the principal is of the form $\tilde{u}(C_2)$ for a decreasing function $\tilde{u}\colon \mathbb{R}^{(\text{time})} \to \mathbb{R}$ if $C_1 \geq S_2$, and $-\infty$ otherwise ($\mathbb{R}^{(\text{time})}$ means the space of time points). The valuation of each agent is simply the minus of his costs. In this case, the principal always communicates only with the agent who is working at the time. Therefore, using the above observation, we can make simplified applications of the following form, with the same values as of the fair applications.

If the principal makes $\tilde{u}$ public at the beginning, then the penalty for the chosen second agent is the loss from the delayed completion of the project, therefore, each second agent should demand $\tilde{u}(C_2) - g_2(S_2)$ money, if he can start his work at S_2 and complete it at C_2. The penalty for the chosen first agent is $-g_2(C_1)$, and each first agent declares how much money he demands for the first task depending on the penalty function. Then the principal chooses the pair by which she gains the highest utility.

Formally, the form of the simplified application for the first agents is a function $g_1\colon (\mathbb{R}^{(\text{time})} \to \mathbb{R}) \to \mathbb{R}$, and for the second agents this is a function $g_2\colon \mathbb{R}^{(\text{time})} \to \mathbb{R}$. If all applications are so, then the principal chooses a pair for which $g_1(g_2)$ is the greatest. Then she tells the penalty function g_2 to the chosen first agent at the beginning of his workflow, and, after his completion, she pays him $g_2(C_1) - g_1(g_2)$. Then the chosen second agent can start his work at C_1 and after his completion, he gets $\tilde{u}(C_2) - g_2(C_1)$. This way, the principal gets utility $\tilde{u}(C_2) - \big(g_2(C_1) - g_1(g_2)\big) - \big(\tilde{u}(C_2) - g_2(C_1)\big) = g_1(g_2)$.

In the simplified fair applications, g_1 and g_2 are chosen in such a way that makes their expected utility independent of the arguments (h and C_2, resp.), if the agents use their best strategies afterwards.

If a first agent has no choice in his workflow, that is, his completion time C_1 is simply a probabilistic variable, then he should choose $g_1(h) = \mathrm{E}\big(h(C_1)\big) - x$, where x is his costs plus his profit.

A.3 Controlling and controlled players

For an example, consider a task of building a unit of railroad. An agent i can make this task for a cost of 100, but with 1% probability of failure which would cause a huge loss of 10,000. Another agent j could inspect and, in the case of failure, correct the work of i under the following conditions. The inspection costs 1. If the task was made correctly, then he does nothing else. If not, he detects and corrects the failure with probability 99%, and for a further cost 100, but he does not detect, therefore, he does nothing with probability 1%. If both of them use cost price strategy and they are the accepted agents for the task, then the mechanism works in the following way.

In the end, i gets 101.99, but pays 199 (totally he pays 97.01) compensation if he fails. j gets 1 if he correctly finds the task to be correct, and, beyond this, he gets 200 if the task was made badly but he corrects it, but he pays 9800 if he misses correcting it.

It can be checked that the expected utility of each agent is his profit independently of the behavior of the others, and that the utility of the principal is fixed.

A.4 Bayesian games under the first price mechanism

We give a rough description of the Bayesian version of our setup called **Bayesian game**. We add a Bayesian rule about the "real information" of the players, namely, nature sends messages to the players including their basic information, and nature makes all its moves, including the choice of the types, governed by a commonly known (but not verifiable) prior. The information of each player should be independent of the future and of the present chance events of others, given his/her extended information. We note that a Bayesian game may even allow sending messages: we should just add actions to the original game with no effect, and choose the strategy of nature so that it tells these actions to other players.

We do not know any Bayesian game when the first price mechanism perform poorly. Therefore, for example, the following questions may be true.

Question 17. *Does there exist a continuous function $f : \mathbb{R} \times \mathbb{R}^{\mathcal{A}g}$ with $f(x, 0, 0, 0, ..., 0) = x$ satisfying that for any Bayesian game B, under the first price mechanism, all undominated subgame-perfect Bayesian Nash equilibria q satisfies*

- $\mathrm{E}\big(u_P(q)\big) \geq f\Big(\mathrm{E}\big(V(B)\big), \mathrm{E}\big(v_i^{\pm}(\theta_i)\big)_{i\in\mathcal{A}g}\Big)$, *or*

- $\mathrm{E}\big(u_P(q)\big) \geq \mathrm{E}\bigg(f\Big(V(B), \big(v_i^{\pm}(\theta_i)\big)_{i\in\mathcal{A}g}\Big)\bigg)$*?*

With dominated strategies, we can easily find a counterexample: if every agent offers to make a money-burning action (as a public consequence), and reports that his valuation is $-\infty$ unless if every agent burns this amount of money, then this can be a weakly dominated subgame-perfect Nash-equilibrium, which is very inefficient.

Although we cannot prove such a statement in general, the following theorem tries to show that under practically reasonable assumptions or approximations, the agents use fair strategies, therefore, the level of inefficiency is no more than in a first price auction.

Consider an arbitrary Bayesian game. Now $\boldsymbol{\theta}$ is a stochastic variable. Let P_i and E_i denote the probability and expectation given the information of $i \in \mathcal{A}g$. From now on, which constants to add to applications are dependent on the information of the agent when he submits his application. The definitions of fair application and fair strategy change correspondingly.

Let us fix $i \in \mathcal{A}g$ and $\boldsymbol{s}_{-i}$. Consider a strategy s_i and denote by $\hat{\theta}_i$ the application he would send by s_i. Let (value) $Val = v_i^{\pm}(\theta_i)$, (difference) $D = Val - v_i^{\pm}(\hat{\theta}_i) = V(\theta_i) - V(\hat{\theta}_i)$ and $e = e(\hat{\theta}_i) = \mathrm{E}_i(D \mid D < Val)$.

In practice, Val and D are almost independent and both have "natural" distributions, therefore, $\mathrm{P}_i(e < Val) = \mathrm{P}\big(\mathrm{E}_i(D \mid D < Val) < Val\big)$ is usually not smaller than $\mathrm{P}_i(D < Val)$. This observation shows the importance of the following theorem.

Theorem 18. *If $\boldsymbol{s}_{-i}$ consists only of fair-like strategies and $\mathrm{P}_i\big(\mathrm{E}_i(D \mid D < Val) < Val\big) \geq \mathrm{P}_i(D < Val)$ holds for all s_i, then a fair strategy of i provides him the highest expected utility.*

Proof. Let $\bar{u}(s_i) = \mathrm{E}_i\big(u_i(s_i)\big)$ and let x be the number by which $\bar{u}(cp_i + x)$ is the largest possible. What we need to prove is that $\forall s_i \colon \bar{u}(cp_i + x) \geq \bar{u}(s_i)$.

Theoretically, let us allow for $i \in \mathcal{A}g$ to submit his fair application with the signed value $v_i^{\pm}(\hat{\theta}'_i)$, namely, submitting $\theta_i + v_i^{\pm}(\theta_i) - v_i^{\pm}(\hat{\theta}'_i)$, for an arbitrary θ'_i chosen by i. Normally, it is an invalid action, because i does not know the values of his possible applications at this time, but for calculation, we shall allow him this, and we denote the fair strategy but with this application by $fair_i(\theta'_i)$.

$\hat{\theta}_i$ is accepted if and only if $v_i^{\pm}(\hat{\theta}_i) > 0$, or equivalently, $D < Val$. By the equation

$$\bar{u}(s_i) = \mathrm{P}_i(i \in \mathcal{A}cc) \cdot \mathrm{E}_i\big(u_i(s_i) \bigm| i \in \mathcal{A}cc\big),$$

we get $\bar{u}(cp_i - e) \overset{(49)}{=} \mathrm{P}_i(e < Val)e$, and $\bar{u}\big(fair_i(\hat{\theta}_i)\big) = \mathrm{P}_i(D < Val)e$, whence we can simply get that

$$\bar{u}(cp_i - x) - \bar{u}(s_i) =$$
$$= \Big(\bar{u}(cp_i - x) - \bar{u}(cp_i - e)\Big) + \Big(\mathrm{P}_i(e < Val) - \mathrm{P}_i(D < Val)\Big)e + \Big(\bar{u}\big(fair_i(\hat{\theta}_i)\big) - \bar{u}(s_i)\Big).$$

- $\bar{u}(cp_i - x) - \bar{u}(cp_i - e) \geq 0$ by the definition of x.
- If $e \leq 0$, then $\bar{u}(s_i) \leq 0 = \bar{u}(cp_i) \leq \bar{u}(cp_i - x)$, therefore, s_i cannot be a better strategy. Assume that $e > 0$. In this case, because of the assumption in the theorem,
$$\big(\mathrm{P}_i(e < Val) - \mathrm{P}_i(D < Val)\big)e \geq 0.$$
- Theorem 12 implies that $\bar{u}\big(fair_i(\hat{\theta}_i)\big) - \bar{u}(s_i) \geq 0$.

To sum up, $\bar{u}(cp_i - x) - \bar{u}(s_i) \geq 0$, which proves the theorem. □

A.5 Sequentially submitted applications

The author could not answer whether, under the second price mechanism, it is necessary to require that the agents report their types at the same time point. Therefore, the following (not well-defined) question is still open.

Question 19. *Is the cost price strategy profile a meaningful equilibrium under the second price mechanism, even if the agents report their types in an order with arbitrary rule?*

A.6 Offer for cooperation

Consider the case when a player a simply wants to make an offer for cooperation with another player c, which c will either accept or reject. Notice that this situation can be modeled by a special case of the first price mechanism, where a is the only agent and c is the principal, and the offer of a is his application.

B Observations for application

B.1 Modifications during the process

In practice, the types can be extremely difficult, therefore, submitting precise fair applications cannot be expected. Hence players can only present a simplified approximation of their type. Generally, such inaccuracies do not significantly worsen the optimality; nevertheless, this loss can be reduced far more with the following observation.

Assume that someone, whose application has been accepted, can refine his type during the process. It would be beneficial to allow him to modify his application correspondingly. The question is: on what conditions?

The answer for us is to allow him to modify his type if he pays the difference between the maximin utility of the principal with the original and the new applications (given her current basic information). Or, equivalently, the contract as a payment function automatically decreases by this difference. It is easy to see that, for an agent, whether and how to modify his application is the same question as which application to submit among the applications with a given value. Consequently, Theorem 13 shows that changing to the agent's true fair application is in his interest.

In the original approach (but adapted as in Section 7.2), it is equivalent to the following. Each possible modification can be handled as a chance node in the original application with 1 and 0 probabilities for continuing with the original and the modified application, respectively. Because at such a chance node, the principal assigns 0 to the branch of not modifying; and to the modification, she assigns the difference between the values of the reported states of the project after and before.

It may happen that at the beginning, it is too costly for some agent to explore the many improbable branches of his decision tree, especially if he does not yet know whether his application will be accepted; but later, it would be worth exploring better the ones that became probable. These kinds of in-process modifications are what we would like to make possible. We show that each player has approximately the same interest as the total interest in the better scheduling of these small modifications.

The expected utility of an agent with an accepted fair application is fixed and for a nearly fair agent, the little modifications of the other applications have negligible effect. As the modifications of each agent have no influence on the utility of the principal and only this negligible influence on the expected utility of other agents, the change in the expected social welfare is essentially the same as the change in the expected utility of this agent. This confirms the above statement.

A very similar argument shows that, under the first price mechanism, if the principal gets some freedom on her actions as in Section 7.2, and she can also refine her type, then she can do such in-process modifications as well, and her interest in doing this is about as much as the total interest of all players.

B.2 Risk-averse agents

Assume that an agent $i \in \mathcal{A}g$ has a strictly monotone **welfare** function $h_i \colon R \to \mathbb{R}$ and he prefers the higher $\mathrm{E}\big(h_i(u_i)\big)$.

We define an application to be **reasonable** in the same way as the fair application with the only difference being that the agent demands

$$h_i^{-1}\Big(h_i\big(-\hat{v}_i(\hat{l}_i, \boldsymbol{c})\big) + \sum_{\chi \in X_i} \delta^*(\chi)\Big).$$

By a reasonable application, in the case of acceptance, the expected welfare of the valuation of the agent is independent of the choices of the principal. If all applications are reasonable, then the utility of the principal remains fixed. If the agent is risk-neutral, then his reasonable application is fair. These are some reasons why reasonable applications work "quite well". We do not state that it is optimal in any sense, but a reasonable application may be better than a fair application in the risk-averse case.

We note that the evaluation of reasonable applications can be much more difficult than of fair applications, but for each agent i, if $h_i(x) = a_i - b_i \cdot e^{-\lambda_i x}$, then a similar recursion works as in Lemmas 1 and 2.

B.3 Necessity of being informed about one's own process

We assumed that none of the chosen players knew anything better about a chance event of any other chosen agent. We show an example that fails this requirement and that makes the mechanism inefficient. Consider two agents i and j that will definitely be accepted. Assume that i believes the probability of an unfavorable event in his work to be 50%, but j knows that the probability is 60%, he knows the estimation of i, and he also knows that at a particular decision node of his, he will be asked to make a move corresponding to this chance event of i. It can be checked that if the application of i is fair, then if j increases the demanded payment in his application of the more probable case by an amount of money and decreases it in the other case by the same amount, then the value of his application remains the same, but this way, j bets $1:1$ with i on an event of 60% probability.

We note that if an agent i has even a small risk that his information might not dominate the information of someone else about i's probabilities, but i wants to participate anyway, then in order to limit such losses, he could rightfully say that a larger bet can only be increased on worse conditions. Submitting a reasonable application with concave welfare function makes something similar, which is another reason to use this.

B.4 Agents with limited liability

This subsection is only a suggestion for the cases when we have some agents with limited liability, and we cannot provide any clear mathematical statement about its level of efficiency.

Our mechanism requires each agent to be able to pay so much money as the maximum possible damage he could have caused. However, in many cases, there may be some agents who cannot satisfy this requirement. Despite this, accepting such an agent i may still be a good decision, if i is reliable to some degree.

To solve this problem, i should find someone who has enough funds, and who takes the responsibility, for example for an appropriate fee. If the agent is reliable to some degree, then he should be able to find such insurer player j. (It can even be the principal, but is considered as another player.) This method may also be useful when i has enough funds, but is very risk-averse.

Here, i and j work similarly as the controlled and controlling parties in Appendix A.3. The differences are that j does not work here, but he has some knowledge about the types (mainly about the type of i). Furthermore, the role of j here can be combined with his role in Appendix A.3, using the modifications in Section 7.2.

B.5 Simplified workflows

In many situations, this mechanism cannot be applied directly because of its complex administrative requirements. For example, consider a supermarket employing many cashiers. Some of the cashiers can go to work whenever they are asked to, while others need to get to know their schedule well in advance. And everyone can be ill, etc. Clearly, it would be unrealistic to ask them to define their future in a decision tree.

However, it might be useful to construct a weaker but simpler class of reported types which require very few and simple communications, but which could still express the preferences and requests quite well. Such a mechanism might still be much more efficient than the naive mechanisms.

We show an example about how the author believes this would work in practice. A cashier who reports a level of permanent availability would get a base salary X, but he may receive messages like "We need you tomorrow form 7:00 to 15:00 for an increased salary Y, but you get Z deduction if you do not come." X, Y and Z are chosen in a fair way, depending on the level of availability the cashier reported. And of course, everyone can report modifications in their availability (e.g. via an online system) with the fair conditions defined by the mechanism.

Another example is about contracting for gas or electricity. In practice, some companies get cheaper electricity if they accept that in the case of problems in the current supply, they may be switched off. But if the companies gave a rough description of their interests about it, and the electric supplier would also make a stochastic dynamic description of the possible problems, then we would be able to make more efficient decisions for the economy. Here, it is reasonable to accept only a small class of possible types which excludes the possibilities of benefiting from collusion. Therefore, second price mechanism may be useful for such problems.

C Notions and notations

For any symbol x, the definition of x_i also defines $\boldsymbol{x}$ as the vector of all meaningful x_i, and x_S as the vector of x_i for all $i \in S$, unless we define them otherwise. If $\boldsymbol{x}$ is a vector of elements and $\boldsymbol{R}$ is a vector of sets, then $\boldsymbol{x} \in \boldsymbol{R}$ means $\forall i\colon x_i \in R_i$. $\boldsymbol{x}_{-j}$ means the vector of all x_i except x_j, and $(y, \boldsymbol{x}_{-j})$ means the vector $\boldsymbol{x}$ by exchanging x_j to y. When there is no ambiguity, we denote $\{e\}$ by e, e.g. we use $S - e$ instead of $S - \{e\}$. The power set of X is denoted by $2^X = \{Y : Y \subseteq X\}$.

A **game** is defined by the set of players, their information, their action sets and their utilities, as follows. There is a given set $\mathcal{P}l^+ = \mathcal{P}l \cup \{nature\}$ of players and nature, and a set $\mathcal{A}$ of actions with $\emptyset \in \mathcal{A}$ indicating the case when the player does nothing, and there is a set $\mathcal{I}$ of information. The action set and the information of each player $y \in \mathcal{P}l$ at each time point $t \in \mathbb{R}$ are defined by the functions $a[y,t]\colon \big(\mathcal{P}l^+ \times (-\infty, t) \to \mathcal{A}\big) \to 2^{\mathcal{A}})$ and $i[y,t]\colon (\mathcal{P}l^+ \times (-\infty, t) \to \mathcal{A}) \to \mathcal{I}$, respectively. These define the strategy set $\mathcal{S}tr_y$ of each player $y \in \mathcal{P}l$ as the set of all functions $s_y\colon \mathcal{I} \to \mathcal{A}$.[2] Finally, the utility of each player $y \in \mathcal{P}l$ is a function $u_y\colon (\mathcal{P}l^+ \times \mathbb{R} \to \mathcal{A}) \to \mathbb{R}$.

Now, the game and the strategy profile uniquely determine the actions $act\colon \mathcal{P}l \times \mathbb{R} \to \mathcal{A}$.

[2] More precisely, the set of all functions $s_y\colon \mathcal{I} \to \mathcal{A}$ satisfying a difficult condition, e.g. the following. For all $act_{-y}\colon (\mathcal{P}l - y) \times \mathbb{R} \to \mathcal{A}$, there exists a unique $act_y\colon \{y\} \times \mathbb{R} \to \mathcal{A}$ satisfying that for all $t \in \mathbb{R}$, $s\big(i[y,t](act_{\mathcal{P}l \times (-\infty,t)})\big) = act_y(t) \in a[y,t](act_{\mathcal{P}l \times (-\infty,t)})$, and for this act_y, $\big\{t \in \mathbb{R}\colon act_y(t) \neq \emptyset\big\}$ is well-ordered.

For each player $y \in \mathcal{P}l$, utility u_y is a function of the types and strategies of all players and the actions of nature. u_y restricted to a domain implied by a condition *con* is denoted by $u_y(con)$, and we use notations such as $u_y(s')$ instead of $u_y(s = s')$, or $u_y(\boldsymbol{\theta'}, s'_y)$ instead of $u_y(\boldsymbol{\theta} = \boldsymbol{\theta'}, s_y = s'_y)$.

Subgame means the game from a certain time point, either including or excluding the current time point, given the previous actions of all players. For example, we can define this game by changing the action sets of all players at all previous time points to the one-element set containing the corresponding action. **State** is a synonym of subgame, but state of the project refers mainly to the starting time point; for example, we say that a state of the project T_1 is *later* than T_2 if T_1 is a subgame of T_2. A state of the project "preceding" or "after" a time point (of an event) means the game from this time point, including or excluding the time point, respectively.

The dependence of any function or relation f on the game G is denoted by the form f^G, but if G is the default game, then we may omit this superindex.

For the sake of lucidity and gender neutrality, we use feminine pronouns for the principal and masculine pronouns for the agents. In notation, i and j refer to an agent and y refers to a pla$y$er.

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