Bertrand-Edgeworth competition with substantial product differentiation

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Abstract

Since Kreps and Scheinkman's seminal article (1983) a large number of papers have analyzed capacity constraints' potential to relax price competition. However, the ensuing literature has assumed that products are either perfect or very close substitutes. Therefore none of the papers has investigated the interaction between capacity constraints and substantial local monopoly power. The aim of the present paper is to shed light on this question using a standard Hotelling setup. The high level of product differentiation results in a variety of equilibrium firm behavior and it generates at least one pure strategy equilibrium for any capacity level. Thus the presence of local monopoly power challenges one of the most general findings about Bertrand-Edgeworth competition: the non-existence of pure strategy equilibria for some capacity levels.

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1 Introduction

The problem of capacity constrained pricing decision in oligopolies has received considerable attention since Kreps and Scheinkman's seminal article (1983). Most of the work in the field of Bertrand-Edgeworth oligopolies focused on the case of homogeneous goods and the capacities' potential impact of relaxing price competition (some recent examples are Acemoglu et al. (2009), De Frutos and Fabra (2011) and Lepore (2012)). However, assuming horizontally differentiated products beside the capacity constraints might lead to nontrivial and sometimes counter-intuitive results. This observation was first articulated by Wauthy (1996). Product differentiation in itself, just like capacity constraints, might be sufficient for firms to avoid the zero profits predicted by the standard Bertrand pricing model. Boccard and Wauthy (2010) investigate exactly this kind of interaction between capacities and Hotelling-type differentiation and find the absence of an equilibrium in pure strategies for intermediate capacity levels. Canoy (1996) also analyzes a similar Bertrand-Edgeworth model although he models product differentiation in a less standard way.

To our best knowledge, all Bertrand-Edgeworth models with differentiated product (apart from Canoy (1996)) make the following simplifying assumption: the transportation cost is so small compared to the consumers' willingness-to-pay that the firms could profitably serve the whole market, even the consumer located at its other extremity. This low level of product differentiation in turn implies that the market is always covered in equilibrium. Therefore these models do not have to take into account the consumers' participation constraints as in equilibrium they are never binding.

In this paper we investigate the interaction between the local monopoly power and the capacities of firms. This interaction has so far been hidden by the overly restrictive assumption of low product differentiation. Our findings about the nature of equilibria are in striking contrast with the results of Boccard and Wauthy (2010). In a comparable setting to ours, they find that for low levels of product differentiation equilibrium in pure strategies does not exist for intermediate capacities. Our main result is the complete characterization of the equilibria for the case of intermediate product differentiation which shows that at least one pure strategy equilibrium exists for any capacity level. We note that this result also holds for the trivial case of high product differentiation when both firms can act as local monopolies without interacting.

It is also worth mentioning that the simplifying assumption of low levels of product differentiation is also prevailing in models of competition in health care markets. Most of this literature that use Hotelling-type product differentiation assumes that the valuation of consumers is large with respect to the transportation costs (see for example Lyon (1999), Gal-Or (1997) and Brekke et al. (2006)). Another example of a model that uses (implicitly) the same assumption is Ishibashi and Kaneko (2008) that describes price and quantity competition in a mixed duopoly.

The paper is organized as follows. Section 2 describes the model, formulates the profit function and identifies the potential equilibrium strategies. Section 3 contains the main result of the paper, the complete characterization of the equilibria. Section 4 discusses the results in the light of the existing literature and Section 5 concludes.

2 The model

2.1 Setting

We analyze a duopoly with firms denoted x and y that produce substitute products for identical marginal cost c. They choose a price p_i $(i \in \{x, y\})$ for one unit of their product. Assume the firms are located on the two extreme points of a unit-length Hotelling-line (x at $\tau = 0$, y at $\tau = 1$) and transportation cost is linear. Moreover, consumers are uniformly distributed along the line but are otherwise identical. They all seek to buy one unit of the product which provides them a gross surplus v. The value of the outside option of not buying the product is normalized to 0. In addition, the firms face rigid capacity constraints k_x, k_y. The size of these capacities as well as the value of the marginal cost are common knowledge. Firms' objective is to maximize their profit by choosing their price.

A consumer located at point τ purchasing from firm x has a net surplus of

$$v - p_x - t \cdot \tau$$

while purchasing from firm y provides her a net surplus of

$$v - p_y - t \cdot (1 - \tau)$$

where t is the per unit transportation cost.

Assumption. Assume $v/t \leq 1.5$, i.e. the products of the firms are substantially different from one another. Furthermore, to get rid of some trivial cases we will assume $1 < v/t \leq 1.5$ and refer to it as *intermediate* level of product differentiation.

Boccard and Wauthy (2010) analyzes a similar setting as ours, the key difference being the level of product differentiation. They restrict their attention to situations in which products are very similar, namely $v/t > 2$ (we can extend their findings to the case of $v/t > 1.5$ as shown later). Below we argue that this simplifying assumption has a surprisingly large impact on the nature of the equilibria, hence extending the analysis to the case of intermediate capacity levels provides new insights into the mechanisms of capacity constrained oligopolies.

2.2 The profit function

Assuming rational consumers the following two constraints are straightforward. The participation constraint (PC) ensures that a consumer located at point τ buys from firm x only if her net surplus derived from this purchase is non-negative:

$$v \geq p_x + t \cdot \tau \tag{PC}$$

The individual rationality constraint (IR) ensures that a consumer located at point τ buys from firm x only if this provides her a net surplus higher than buying from the competitor:

$$v - p_x - t \cdot \tau \geq v - p_y - t \cdot (1 - \tau) \tag{IR}$$

Let T_x be the marginal consumer who is indifferent whether to buy from firm x or not. In the absence of capacity constraints it is easy to see that T_x is the minimum of the solutions of the binding constraints (PC) and (IR). Let $\overline{T}_x$ be the consumer for whom both of the above constraints are binding. Thus this consumer is indifferent among buying form x, buying from y and not buying at all. The net surplus being decreasing in the distance from firm x implies that (PC) is binding for $T_x \leq \overline{T}_x$ and (IR) is binding if $T_x \geq \overline{T}_x$. (Symmetric formulas apply to firm y.) Hence we know that in case capacities are abundant,

$$p_x = \begin{cases} v - t \cdot T_x & \text{if } T_x \leq \overline{T}_x, \\ p_y + t - 2 \cdot t \cdot T_x & \text{if } T_x \geq \overline{T}_x. \end{cases} \tag{1}$$

Naturally, the existence of capacity constraint means for firm x that it cannot serve more than k_x consumers. We assume that after each consumer chooses the firm to buy from (or not to buy), firms have the possibility to select which consumers to serve and they serve those who are the closest to them. In our setting this corresponds to the assumption of efficient rationing rule, which is extensively used in the literature. Therefore the additional constraints caused by the fixed capacity levels can be written as:

$$T_x \leq k_x \quad \text{and} \quad 1 - T_y \leq k_y \tag{CC}$$

It is important to notice that in some cases, when firm y is capacity constrained, firm x can extract a higher surplus from some consumers by knowing that they cannot purchase from the rival even if they wanted to since firm y does not serve them. Practically, this means that the participation constraint (PC) will always be binding on $\left[\overline{T}_x, 1 - k_y\right]$ whenever this interval is not empty, i.e. whenever the rival's capacity is sufficiently small: $k_y \leq 1 - \overline{T}_x$. Using this observation, one can reformulate (1) for any capacity level:

$$p_x = \begin{cases} v - t \cdot T_x & \text{if } T_x \leq \max\{\overline{T}_x, 1 - k_y\} \ , \\ p_y + t - 2 \cdot t \cdot T_x & \text{if } T_x > \max\{\overline{T}_x, 1 - k_y\} \end{cases} \tag{2}$$

Firms' profit can be simply written as:

$$\pi_x = (p_x - c) \cdot T_x \tag{3}$$

Given the competitor's capacity and its price choice, determining the unit price p_x is equivalent to determining the marginal consumer T_x. The observation that prices and quantities can be used interchangeably will simplify the solution of the model, this technique is also used by Yin (2004).

(3) can thus be rewritten as

$$\pi_x(T_x) = \begin{cases} \pi_x^{LM} = (v - t \cdot T_x) \cdot T_x & \text{if } T_x \leq \max\{\overline{T}_x, 1 - k_y\}, \\ \pi_x^{C} = (p_y + t - 2 \cdot t \cdot T_x) \cdot T_x & \text{if } T_x > \max\{\overline{T}_x, 1 - k_y\} \end{cases} \quad (4)$$

The optimization problem of the firm consists of finding the value T_x which maximizes the above expression satisfying the capacity constraint (CC). The superscript LM stands for Local Monopoly because the firm extracts all the consumer surplus from the marginal consumer when (PC) binds. Similarly, the superscript C stands for Competition since the marginal consumer is indifferent between the offer of the two firms whenever (IR) binds.

2.3 Potential equilibrium strategies

Define $T_x^{LM} = \arg\max_{T_x} \pi_x^{LM}$ and $T_x^C = \arg\max_{T_x} \pi_x^C$, the values at which the two quadratic curves attain their maxima, hence they are local maxima of the profit function $\pi_x(T_x)$.

The relative order of the five variables

$$T_x^{LM}, \quad T_x^C, \quad \overline{T}_x, \quad 1 - k_y \quad \text{and} \quad k_x$$

is crucial in solving the maximization problem. The main difficulty in the solution of the firms' maximization program is twofold. On the one hand, the profit function is discontinuous whenever $k_y \leq 1 - \overline{T}_x$ and kinked otherwise. On the other hand, the values

$$\overline{T}_x = \frac{p_y - v + t}{t} \quad \text{and} \quad T_x^C = \frac{p_y + t}{4t}$$

depend on the choice of the other firm, p_y. The following lemma simplifies the solution considerably.

Lemma 1.
$T_x^{LM} \leq \overline{T}_x$ *implies* $T_x^C \leq \overline{T}_x$ *and* $T_x^C \geq \overline{T}_x$ *implies* $T_x^{LM} \geq T_x^C \geq \overline{T}_x$.

The proof of the lemma is relegated to the Appendix. The form of firm x's profit function hinges on the relative order of $\overline{T}_x$ and $1 - k_y$. Therefore in the following discussion we will separate two cases: In Case A the capacity of firm y is relatively large, $1 - k_y < \overline{T}_x$. In Case B $1 - k_y \geq \overline{T}_x$ which means that firm x may be able to take advantage of the fact that its adversary is relatively capacity constrained.

Case A: $1 - k_y < \overline{T}_x$**.** When the capacity of firm y is relatively large, (1) shows the relation between the price p_x charged by firm x and its demand (captured by the marginal consumer T_x). Using Lemma 1 three different subcases can be identified depending on the parameter values of the model and the competitor's choice.

Lemma 2. *Assume* $1 - k_y < \overline{T}_x$.

(A1) if $T_x^{LM} \leq \overline{T}_x$ *then the optimal choice of firm x is* $\min(T_x^{LM}, k_x)$,

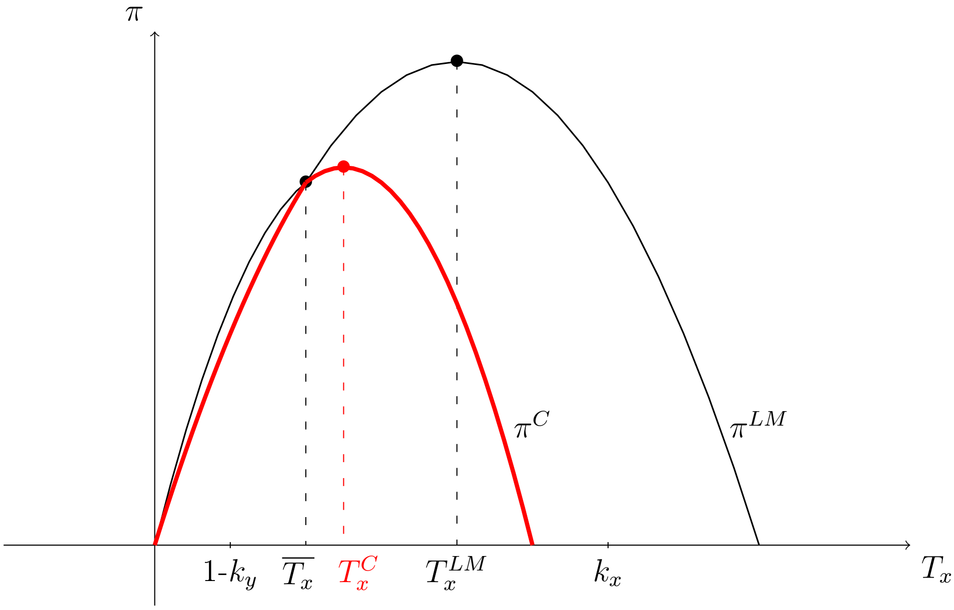
(A2) if $T_x^C \geq \overline{T}_x$ *then the optimal choice of firm x is* $\min(T_x^C, k_x)$,

(A3) if $T_x^C \leq \overline{T}_x \leq T_x^{LM}$ *then the optimal choice of firm x is* $\min(\overline{T}_x, k_x)$.

Considering Lemma 1 it is easy to see that cases A1, A2 and A3 provide a complete partitioning of Case A. Hence for any parameter values in Case 1 and for every possible behavior of the competitor, the lemma identifies the best response strategy of firm x. Symmetric formulas apply for firm y. The complete proof of this lemma is relegated to the Appendix.

However, for an intuition, first notice that the two branches of the profit function, π_x^{LM} and π_x^C are both quadratic functions of T_x that by definition cross each other at 0 and at $\overline{T}_x$. Then depending on the values t, v and T_y one of the three possibilities above will hold. As an illustration of Case A2 when $T_x^C < k_x$ see Figure 1. Using Lemma 1 the condition of the case $T_x^C \geq \overline{T}_x$ immediately implies $T_x^{LM} \geq \overline{T}_x$. We know that the profit function is composed of the function π_x^{LM} on the interval $[0, \overline{T}_x]$ then it switches to function π_x^C . The actual profit

Figure 1: Illustration of Case A2 ($T_x^C < k_x$)



function is thus the thick (red) curve in the figure. Then using the figure it is straightforward to find the optimal choice of firm x. Since the two quadratic and concave functions cross each other before either of them reaches its maximum, the maximal profit will be attained on the second segment where $\pi_x = \pi_x^C$. By definition, $\arg\max_{T_x} \pi_x^C = T_x^C$ is the optimal choice, and the assumption $T_x^C < k_x$ makes this feasible.

Case B: $\overline{T}_x \leq 1 - k_y$. In Case B, the rival of firm x disposes of relatively low capacity. Therefore firm x might be inclined to take advantage of the fact that firm y is not capable of serving consumers located on the interval $[0, 1 - k_y]$. On this segment firm x does not have to care about its competitor's price and the individual rationality constraint (IR), it is only threatened by some consumers choosing the outside option of not buying the product (PC) and eventually by its own capacity constraint.

Lemma 3. *Assume $\overline{T}_x \leq 1 - k_y$. Then*

(B1) if $T_x^{LM} \leq \overline{T}_x$ then the optimal choice of firm x is $\min(T_x^{LM}, k_x)$,

(B2) if $\overline{T}_x \leq T_x^C \leq 1 - k_y$ then the optimal choice of firm x is $\min(1 - k_y, T_x^{LM}, k_x)$,

(B3) if $\overline{T}_x \leq 1 - k_y \leq T_x^C$ *then the optimal choice of firm x is either* $\min(1 - k_y, k_x)$ *or* $\min(T_x^C, k_x)$*,*

(B4) if $T_x^C \leq \overline{T}_x \leq 1 - k_y \leq T_x^{LM}$ *then the optimal choice of firm x is* $\min(1 - k_y, k_x)$*.*

(B5) if $T_x^C \leq \overline{T}_x \leq T_x^{LM} \leq 1 - k_y$ *then the optimal choice of firm x is* $\min(T_x^{LM}, k_x)$*.*

Notice that case B1 corresponds exactly to case A1 of Lemma 2 and B5 also describes a very similar situation. However, the other cases are affected by the limited capacity of the rival firm. The case closest to case A2 pictured above is case B2. The only difference is in the size of the rival firm's capacity, here it is assumed to be much smaller. As an illustration of this situation, see Figure 2 (where we assumed k_x large in order to draw a clearer picture). As is clear from the figure and true in general, $\pi_x^{LM}(\tau) > \pi_x^C(\tau)$ whenever $\tau > \overline{T}_x$ i.e. to the right of the crossing point of the two curves. Hence the profit function is not only non-differentiable as in the above case, it is also discontinuous at $1 - k_y$. Therefore the assumption $T_x^C \leq 1 - k_y \leq T_x^{LM}$ immediately implies that $1 - k_y$ is the optimal choice of firm x, i.e. it produces up to the capacity of the other firm. The profit curve and the optimal solution are shown in thick (red) on Figure 2.

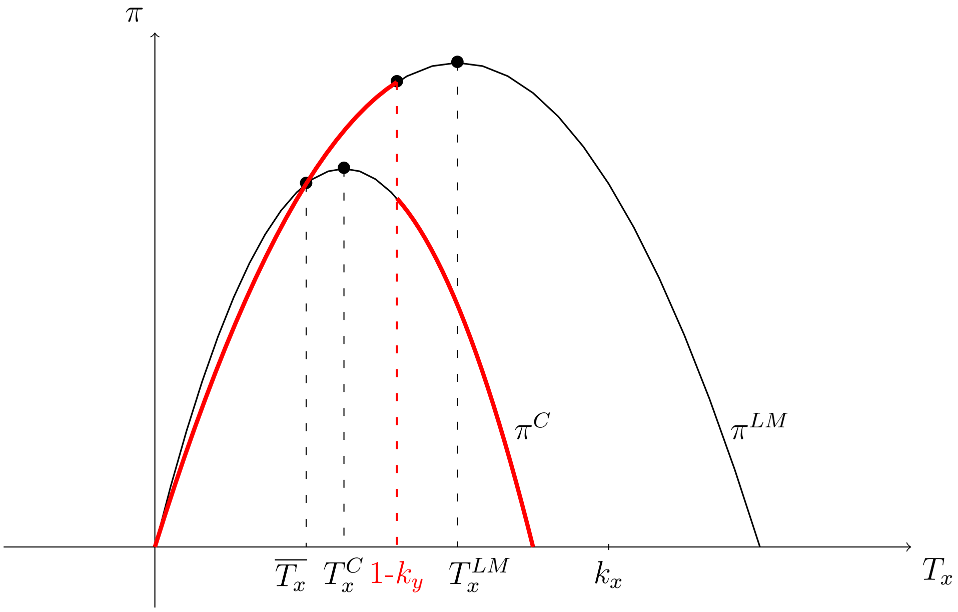
The most interesting case is arguably B3 where 3 different best replies may arise depending on the exact parameters of the model and the competitor's choice. This is also the most problematic case in Boccard and Wauthy (2010) in the sense that this discontinuity inhibits the possible existence of pure strategy equilibrium. As we will show below, case B3 never arises in equilibrium when assuming intermediate levels of product differentiation.

The next section describes the numerous equilibria of the game using the conditional best replies of firms described above.

3 Equilibria

In this section we will determine which kinds of equilibria may arise in the intermediate product differentiation case as a function of firms' capacities and the other parameters of the model (v and t). The calculations will be based

Figure 2: Illustration of Case B2 ($1 - k_y < T_x^{LM} < k_x$)



on the results of Lemmas 2 and 3 that describe the firms' conditional best responses.

As is clear from those lemmas, there are 5 potential equilibrium strategies for both firms:

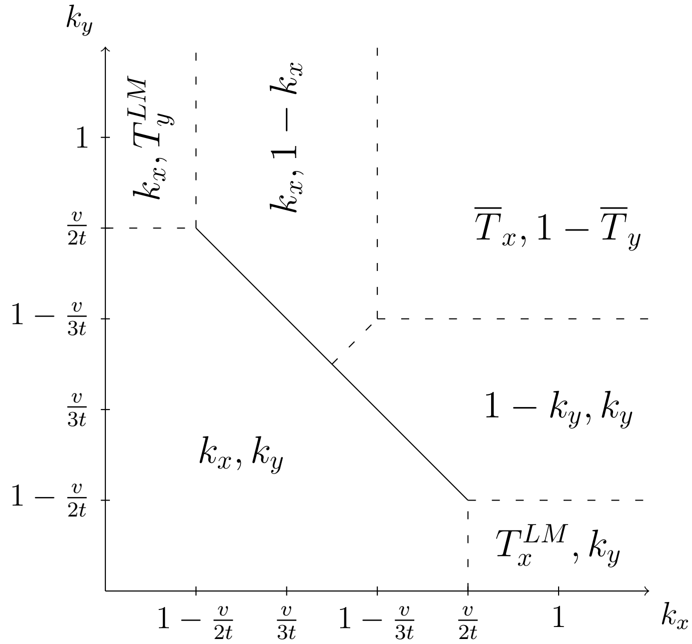
$$T_x^{LM}, \quad T_x^C, \quad \overline{T}_x, \quad 1 - k_y \quad \text{and} \quad k_x$$

The exercise of finding all equilibria consists of comparing the conditions for potential equilibrium strategies (described in cases A1-A3 and B1-B5) of firm x to those of firm y one-by-one and determining whether the conditions are compatible. In case they are, we also have to formulate the conditions in terms of the parameters of the model. Since the cases described in the two lemmas are exhaustive, this method finds all the existing equilibria of the game. These case-by-case calculations are by nature tedious so we relegate them to the Appendix. The following proposition summarizes the main result of the paper.

Proposition 1. *For intermediate levels of product differentiation, i.e. for* $1 < v/t \leq 1.5$ *there exists at least one equilibrium in pure strategies for any capacity pair* (k_x, k_y)*. The nature of the equilibria depends on the relative size of the capacity levels, and the relative value of consumers' willingness-to-pay* v *and their transportation cost* t*.*

Proposition 1 is in striking contrast to most of the existing results about

Figure 3: Equilibria with substantial product differentiation ($1 < v/t \leq 1.2$)



Bertrand-Edgeworth oligopolies. The usual finding in the existing literature is that there is at least one region of capacity levels for which there does not exist a pure strategy equilibrium. This clearly shows that the presence of substantial local monopoly power changes Bertrand-Edgeworth competition drastically. Even Boccard and Wauthy (2010) who investigate the case of slightly differentiated products face the problem of non-existence of pure strategy equilibrium, indeed, their main contribution is a partial characterization of the mixed strategy equilibrium.

We provide a complete characterization of the equilibria of our model. Figure 3 illustrates the different types of equilibria that arise as a function of the parameters. For simplicity the figure depicts only the case of $1 < v/t \leq 1.2$. (The complement case of $1.2 < v/t \leq 1.5$ is qualitatively equivalent, the same type of equilibria arise, the only difference is in the ordering of the different values on the axes.)

The capacities of firm x and y are shown on the horizontal and the vertical axis, respectively. The values written in every parameter region show the equilibrium strategy of firm x and y, respectively. Note that the figure is symmetric which is sensible since apart from their capacities the firms are identical.

Capacity constrained equilibria The simplest case is the one where k_x and k_y are both very low ($k_x + k_y < 1$) which inhibits the interaction between the two firms. Consequently they maximize their profits independently by producing up to their capacity. Therefore (k_x, k_y) is the unique equilibrium in this region. Assuming a similarly low capacity for firm y ($k_y < 1 - \frac{v}{2t}$) but a larger one for firm x ($k_x \geq \frac{v}{2t}$), one gets to the region where firm x cannot profitably increase its production and implements its unconstrained local monopoly profit $T_x^{LM} = \frac{v}{2t}$. Hence (T_x^{LM}, k_y) is the unique equilibrium here.

Capacity constrained secret handshake equilibria The most interesting region is arguably the one where the capacity of one firm is not very low but not very high either ($\max(1 - \frac{v}{2t}, \frac{v}{3t}) < k_y < \min(1 - \frac{v}{3t}, \frac{v}{2t})$) and the industry capacity is sufficient to cover the market ($k_x + k_y \geq 1$). Firm y producing up to its capacity and firm x deciding to serve the remaining $1 - k_y$ consumers is a pure strategy equilibrium of this region. Notice that the size of their capacity would allow firms to enter into direct competition, however, it would not be profitable for firm x. Instead it prefers to match the residual demand of the market. Essegaier et al. (2002) find similar equilibrium behavior in their model with heterogeneous demand and call it a "secret handshake" equilibrium. Notice that in the triangle-shaped region $k_x, k_y < \min(1 - \frac{v}{3t}, \frac{v}{2t})$ and $k_x + k_y \geq 1$ either firm producing up to its capacity with the other one engaging in the secret handshake constitutes an equilibrium. Thus in this region 2 pure strategy equilibria coexist with mixed strategy equilibria.

Unconstrained secret handshake equilibria Lastly, when both capacities are large ($k_x, k_y > \min(1 - \frac{v}{3t}, \frac{v}{2t})$) there is a continuum of equilibria in pure strategies. As $\overline{T}_x$ depends on p_y and thus on T_y and vice versa, the location of the indifferent consumer ($\overline{T}_x = 1 - \overline{T}_y$) may take any values in between $\max(1 - \frac{v}{2t}, \frac{v}{3t})$ and $\min(1 - \frac{v}{3t}, \frac{v}{2t})$. Furthermore, these equilibria could also be described as a type of secret handshake since here $\overline{T}_x + \overline{T}_y = 1$ holds so the market is exactly covered by the two firms. We also note that the multiplicity of equilibria is a standard result for Hotelling models with substantial product differentiation without capacity constraints, so its presence is natural for the case of abundant capacities.

4 Discussion

To see how our results are related to the existing literature, it is worthwhile comparing the case of intermediate capacity levels with varying degrees of product differentiation:

i $v/t = \infty$: mixed strategy equilibria with continuous support

ii $2 < v/t < \infty$: mixed strategy equilibria with finite support

iii $1.5 < v/t \leq 2$: mixed strategy equilibria with a support consisting of 2 strategies

iv $1 < v/t \leq 1.5$: nontrivial pure strategy equilibrium

v $v/t \leq 1$: trivial pure strategy equilibrium

(i) is the case of homogeneous goods which is the seminal result of Kreps and Scheinkman (1983). (ii) is the main result of Boccard and Wauthy (2010). Furthermore, they prove that the number of atoms used in equilibrium is decreasing in v/t. (iii) is the extension of this result, when $1.5 < v/t \leq 2$ we can show that one firm uses a pure strategy to keep the other firm indifferent between choosing 2 prices, hence this firm uses a mixed strategy. (iv) is our main result, for intermediate capacity levels the number of atoms used in equilibrium reduces to 1. Although (v) is trivial, it respects the continuity of the evolution of the equilibrium as a function of the degree of product differentiation.

5 Conclusion

We analyze a Bertrand-Edgeworth duopoly with exogenous capacity constraints and a non-negligible degree of product differentiation. The complete characterization of the model's equilibria was feasible and showed that there exists at least one pure strategy equilibrium for any capacity level. This contrasts with the usual result of existing Bertrand-Edgeworth models that find the nonexistence of such equilibria for some capacity levels. Thus our main finding illuminates the importance of local monopoly power in the price setting of capacity constrained industries.

Appendix

Proof of Lemma 2 It is easy to see that

$$T_x^{LM} = \frac{v}{2t}, \quad \overline{T}_x = \frac{p_y - v + t}{t} \quad \text{and} \quad T_x^C = \frac{p_y + t}{4t}.$$

Then for any $t > 0$

$$T_x^{LM} \leq \overline{T}_x \iff \frac{v}{2t} \leq \frac{p_y - v + t}{t} \iff p_y \geq \frac{3}{2}v - t$$

and similarly

$$T_x^C \leq \overline{T}_x \iff \frac{p_y + t}{4t} \leq \frac{p_y - v + t}{t} \iff p_y \geq \frac{4}{3}v - t$$

also

$$T_x^{LM} \leq T_x^C \iff \frac{v}{2t} \leq \frac{p_y + t}{4t} \iff p_y \geq 2v - t$$

This proves the two parts of the lemma for any $v > 0$.

Proof of Lemma 3

(A1) First assume $T_x^{LM} < k_x$. By Lemma 2 the condition $T_x^{LM} < \overline{T}_x$ implies $T_x^C < \overline{T}_x$. By definition T_x^{LM} is the profit maximizing quantity on the π_x^{LM} curve. Hence

$$\pi_x^{LM}(T_x^{LM}) \geq \pi_x^{LM}(\overline{T}_x) = \pi_x^C(\overline{T}_x) \geq \pi_x^C(\tau) \quad \text{for all } \tau > \overline{T}_x$$

where the last inequality holds because $T_x^C < \overline{T}_x$ means that π_x^C is decreasing on the interval in question.

k_x is clearly the optimal choice when $T_x^{LM} \geq k_x$ as π_x^{LM} is increasing up to T_x^{LM}.

(A2) is proved in the main text.

(A3) Assume $\overline{T}_x < k_x$. Firstly, $T_x^C \leq \overline{T}_x$ implies that

$$\pi_x^{LM}(\overline{T}_x) = \pi_x^C(\overline{T}_x) \geq \pi_x^C(\tau) \quad \text{for all } \tau > \overline{T}_x$$

Secondly, $\overline{T}_x \leq T_x^{LM}$ implies that

$$\pi_x^{LM}(\tau) \leq \pi_x^{LM}(\overline{T}_x) = \pi_x^C(\overline{T}_x) \quad \text{for all } \tau < \overline{T}_x$$

This means that the profit function is increasing up to $\overline{T}_x$ and then it is decreasing. Again, k_x is clearly the optimal choice when $\overline{T}_x \geq k_x$ as π_x^{LM} is increasing up to $\overline{T}_x$.

□

Proof of Lemma 4

(B1) The proof of case (B1) is identical to the proof of case (A1) above.

(B2) is proved in the main text.

(B3) $\overline{T}_x \leq 1 - k_y \leq T_x^C$ implies that firm x must compare $\pi_x^{LM}(1-k_y)$ to $\pi_x^C(T_x^C)$ which are the two local maxima of the profit function, except if k_x is low, then the capacity might be the optimal choice.

(B4) Given the condition $\overline{T}_x < 1 - k_y$, the constraint (PC) binds on $[0, 1-k_y]$. The profit function π_x^{LM} is increasing up to $1-k_y$ since $T_x^{LM} > 1-k_y$. Moreover, $\pi_x^{LM}(1-k_y) > \pi_x^C(1-k_y)$ and also π_x^C is decreasing above $1-k_y$.

(B5) Given the condition $\overline{T}_x < 1 - k_y$, the constraint (PC) binds on $[0, 1-k_y]$. The unconstrained optimum at $T_x^{LM} (< 1-k_y)$ is feasible for x whenever its capacity is sufficiently large.

□

Proof of Proposition 1 The proof builds heavily on the results of Lemmas 3 and 4 that identify parameter regions in which one of the 5 potential equilibrium strategies dominate any other strategy for a given firm. In the following we check the conditions of the 15 possible combinations of the potentially dominating strategies of the two firms and determine whether they are compatible or not.

Firstly, notice that any case where $k_x + k_y \leq 1$ is trivial: the firms do not have sufficient capacity to cover the market, they can never enter into competition. Hence

$\pi_i = \pi_i^{LM}$ and the only possible equilibrium is both firms playing $\min(T_i^{LM}, k_i)$.

Consider the 5 cases in which firm x plays T_x^{LM}:

T_y^{LM}: When firm y plays T_y^{LM} both firms play $v/2t$ and their price is equal to $p_x = p_y = v/2$. This may only happen if the conditions of (A1) or (B1) are satisfied for both firm. Those conditions imply $e_i > \frac{3}{2}v - t$ which in turn implies $v/t < 1$ which contradicts our main assumption of intermediate degree of product differentiation. Therefore this case will never arise in equilibrium.

T_y^C: Firm x playing T_x^{LM} while firm y plays T_y^C can never happen since by definition this would entail (IR) binding for firm y and slack for firm x which is a contradiction.

$\overline{T}_y$: Firm y cannot play $\overline{T}_y$ for the same reason it cannot play T_y^C.

$1 - k_x$: Firm y playing $1 - k_x$ is incompatible with x playing T_x^{LM}. Notice that the latter induces

$$\frac{v}{2t} < k_x \iff 1 - k_x < 1 - \frac{v}{2t} = \overline{T}_y$$

where the last equality follows from $p_x = v/2$. But the inequality above contradicts with (B2), (B3) and (B4) so $1 - k_x$ can never be optimal for firm y.

k_y: Firm y playing k_y is the only case that arises in equilibrium when firm x plays T_x^{LM}. Notice that $p_x = v/2$ and $p_y = v - t \cdot k_y$. The optimality conditions imply $k_y < 1 - v/2t$. Also, it is easy to see that

$$1 - T_y^C < 1 - \overline{T}_y < 1 - T_y^{LM}$$

which means by Lemma 4 that y should play $\min(\overline{T}_y, k_y)$. Since $\overline{T}_y = 1 - v/2t$ it is indeed optimal for firm y to play k_y.

The conditions for a (T_x^{LM}, k_y)-type equilibrium are hence the following: $k_x > v/2t$ and $k_y < 1 - v/2t$. Notice that these are exactly the conditions required for case (B5).

Now consider the 4 cases where firm x plays $1-k_y$. (The remaining fifth such case is symmetric to one case analyzed above.) This may only be optimal for the firm if one of the conditions (B2), (B3) or (B4) holds. Notice that it is common among these conditions that $\overline{T}_x \leq 1-k_y$, moreover, $1-k_y$ is only played when (PC) binds so $p_x = v - t\cdot(1-k_y)$.

k_y: If firm y plays k_y, $p_y = v - t\cdot k_y$ always holds. Conditions for (B2) imply $p_y < \frac{4}{3}v - t$ and $T_x^C < 1-k_y$ which imply $1 - v/3t < k_y < 1 - v/3t$ so (B2) is not compatible with k_y.

Conditions for (B3) require that $\pi_x^{LM}(1-k_y) > \pi_x^C(T_x^C)$ which is equivalent to

$$0 > \frac{(v+t(1-k_y))^2}{8t} - (v-(1-k_y)(1-k_y)) \iff 0 > [v-3t(1-k_y)]^2$$

which is impossible, so (B3) is also incompatible with k_y.

Conditions for (B4) are in turn compatible with y playing k_y. The conditions for a $(1-k_y, k_y)$-type equilibrium are the following:

$$\max(1-\frac{v}{2t}, \frac{v}{3t}) < k_y < \min(1-\frac{v}{3t}, \frac{v}{2t}) \quad \text{and} \quad k_x + k_y > 1.$$

$\overline{T}_y$: Notice that when firm y plays $\overline{T}_y$ and firm x plays $1-k_y$, $\overline{T}_y = k_y$ so the cut-off value for firm y exactly coincides with its capacity. This means that this case is identical to the one above.

T_y^C: Notice that T_y^C is only played by firm y if $T_y^C > \overline{T}_y$ which implies $p_x < \frac{4}{3}v - t$ which is equivalent to $k_y < v/3t$. However, $T_y^C < k_y$ which entails $k_y > v/3t$ is also necessary. This shows that T_y^C is incompatible with firm x playing $1-k_y$.

$1-k_x$: Firm y playing $1-k_x$ is incompatible with x playing $1-k_y$. Notice that $\overline{T}_y = k_y$ and $\overline{T}_x = k_x$. Moreover, the optimality of these strategies requires $\overline{T}_x \leq 1-k_x$ and $\overline{T}_y \leq 1-k_y$ which then entails $k_x, k_y \leq 1/2$ which is impossible.

Now consider the 3 cases when firm x plays $\overline{T}_x$.

$\overline{T}_y$: Notice that when firm y plays $\overline{T}_y$ and firm x plays $\overline{T}_x$, the conditions of optimality translate to $p_x + p_y = 2v - t$ and also $\frac{4}{3}v - t < p_y < \frac{3}{2}v - t$.

Furthermore, conditions concerning the capacities require $k_x, k_y \geq \min(1 - \frac{v}{3t}, \frac{v}{2t})$.

k_y: Firm y playing k_y and firm x playing $\overline{T}_x$ is possible only if $k_y = \overline{T}_y$ otherwise the (IR) constraint would bind for the one firm but not for the other. If this is true, the case is naturally identical to the case above.

T_y^C: Firm y playing T_y^C is impossible when firm plays $\overline{T}_x$ because then the constraint (IR) would be binding for firm x and slack for firm y which is a contradiction.

Now consider the 2 cases when firm x plays T_x^C.

T_y^C: Both firms playing the competitive strategy leads to $p_x = p_y = t$ and both firms serving exactly 1/2 of the market. However, this requires product differentiation to be low, $v/t > 1.5$ which case is not the object of the present paper.

k_y: Firm y playing k_y and firm x playing T_y^C is possible only if $k_y = T_y^C$ otherwise the (IR) constraint would bind for the one firm but not for the other. If this is true, the case is naturally identical to the case above.

The remaining case is when both firms play up to their capacity. Of course this is impossible when $k_x + k_y > 1$. Otherwise the (k_x, k_y)-type equilibrium is played which is already described above.

□

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