

# Financially constrained strategic trading and endogenous predation in illiquid markets

Sitikantha Parida

London School of Economics

Gyorgy Venter\*

London School of Economics

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## Abstract

In this paper we develop a model of strategic trading which explicitly allows for forced liquidation. Large agents hold an illiquid risky security and have to satisfy minimum capital requirements: when a trader's portfolio wealth goes below a certain threshold, she is forced to unwind her total risky asset positions immediately. Strategic traders with significant price impact take this constraint into account and try to benefit from fire sales of others. The model has a very tractable linear structure and shows that the behaviour of large traders depends on their exposure to fluctuations in the price of the risky asset: when traders have similar proportion of wealth invested in the risky security, they behave cooperatively and spread their orders over several trading periods, while if there is significant difference in this ratio, the stronger agent (with low proportion of wealth invested in the asset) predates on the weak trader and forces it to exit the market.

## 1 Introduction

Large traders, such as dealers, mutual funds and pension funds, play an important role in financial markets. Several empirical studies demonstrate that these institutional investors' trades have significant price impact as their strategies often involve dealing with large positions in assets held by a relatively few number of investors. Recent studies have also shown that large agents strategically reduce this price impact of their trades by spreading them over several days. When there are a finite number of large traders (but at least two), strategic traders do not just account for their own price impact when making

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\*corresponding author; email: g.venter@lse.ac.uk.

portfolio choice decisions, but also for the impact of their trades on portfolio decisions of other market participants.

These institutional investors are usually subject to regulation, for example, they face short-sale or borrowing constraints, which we call exogenous frictions, or have to trade on margins or use risk-management policies (e.g. Value-at-Risk). The second group of these trading frictions are endogenous in the sense that 1. they depend on market prices and thus imply that institutional traders can influence the dynamics of prices on which the capital requirements are based, and 2. violation of the constraint implies retrospective action and may require partial or full liquidation. The effect of these constraints is hence twofold: on one side, large agents subject to endogenous constraints will reduce their trading speed in order not to violate the constraints, but on the other side, they might have an important predatory motive to push other institutional traders to distress and force them to leave the market.

This phenomena is well documented in the popular press: One recent example of a forced liquidation is that of Focus Capital. According to an article of Financial Times written in March 2008:

"In a letter to investors, the founders of Focus, Tim O'Brien and Philippe Bubb, said it had been hit by "violent short-selling by other market participants", which accelerated when rumors that it was in trouble circulated. Sharp drops in the value of its investments led its two main banks to force it to sell last Tuesday, according to the letter." FT 04/03/08.

Another famous example of predatory trading is when Goldman Sachs & Co. and other counterparties traded against LTCM in 1998. The proposal of UBS Warburg, to take over Enron's traders without taking over its trading positions, was opposed on the same ground - it presented potential predatory risk. There have been also evidence of predatory trading during 1987 stock market crash (Brady et al., 1988).

In this paper we study a dynamic model of strategic trading which explicitly allows for forced liquidations. We propose a multiperiod model with large strategic traders. Strategic traders hold an illiquid asset and face leverage or risk management constraints. Agents have a margin account for the risky asset and a position in the riskless asset, and the overall position in this account has to be such that the account's value remains positive. When the portfolio value becomes negative, the trader gets a margin call, and has to unwind her total risky position immediately and leave the market. This margin account implies that arbitrageurs' wealth limits the positions they can take as long as they do not want to violate the constraint. Other strategic traders consider this forced liquidation and may trigger the distress of 'weaker' agents. We show that the behaviour of large traders depends on their exposure to fluctuations in the price of the risky asset.

When traders have similar proportion of wealth invested in the risky security, they behave cooperatively and spread their orders over several trading periods, as in a benchmark model without the constraint. However, if there is a significant difference in the ratio of risky and total wealth between the two traders, the 'strong' agent (with low proportion of wealth invested in the asset) predates on the 'weak' trader (with high proportion of wealth invested in the asset) by manipulating the price in the first period, and forces her to unwind her risky position at a large discount. By doing this the strong agents benefits from the fire sale resulting from the forced liquidation of the weak agent.

Related literature:

- Attari and Mello [1]
- Shleifer and Vishny [5]
- Gromb and Vayanos [4]
- Brunnermeier and Pedersen [3]
- Carlin, Lobo, Viswanathan
- Fardeau.

## 2 The model

In this section we first describe the setup of the economy, we introduce strategic traders and the financial constraints they.

The economy consists of two trading periods (0 and 1) and two types of agents – value-based traders and strategic agents.

### 2.1 Assets

There are two assets. The first is a one period risk-free asset that pays a constant gross return of  $R$  which we normalize to  $R = 1$ . The second is a long-lived risky asset that pays a dividend of  $d$  at the end of period 1, where  $d$  is normally distributed with mean  $\bar{d}$  and variance  $\sigma^2$ . Let  $p_t$  denote the market price of the risky asset at time  $t = 0$  and 1.

### 2.2 Agents

In each period agents trade with each other by submitting trade orders which clear the market in a Walrasian framework. There are two types of traders present in the model: long-term traders and strategic traders.

### 2.2.1 Long-term traders

We assume a competitive fringe consisting of 'long-term' or 'value-based' traders as follows. Some agents, 0-investors, enter the market at date 0, trade, exit the market and hold their portfolio until the final payoff of the risky asset. Similarly, 1-investors enter the market at date 1, trade and hold their portfolio until the payoff of the risky asset.

The  $t$ -investors,  $t = 0, 1$ , are competitive, form a continuum of measure 1, and have initial wealth  $\bar{w}^t$ . They choose holdings of the risky asset,  $y_t$ , to maximize the expected utility of final wealth. We model long-term traders to be risk-averse with CARA-coefficient  $\bar{\alpha}$ . Their optimization problem is

$$\max_{y_t} -E \left[ \exp \left( -\bar{\alpha} w_1^t \right) \right],$$

subject to the budget constraint  $\bar{w}_1^t = \bar{w}^t - p_t y_t + d y_t$ .

Long-term traders have CARA-utility on their normally distributed final wealth, hence their optimization problem is equivalent to:

$$\max_{y_t} CE \left( \bar{w}_1^t \right) = \bar{w}^t - p_t y_t + \bar{d} y_t - \frac{\bar{\alpha}}{2} \sigma^2 y_t^2,$$

which yields that the optimal demand of  $t$ -investors is

$$y_t(p_t) = \frac{1}{\bar{\alpha} \sigma^2} (\bar{d} - p_t) = \frac{1}{\lambda} (\bar{d} - p_t) \quad (1)$$

in periods  $t = 0$  and 1. As our focus will be on the behaviour of strategic agents, we can introduce the parameter  $\lambda \equiv \bar{\alpha} \sigma^2 \geq 0$ , and note that from the viewpoint of strategic traders  $\lambda$  represents market illiquidity (or  $\frac{1}{\lambda}$  is market depth). Equation 1 shows that  $t$ -investors will purchase the asset if  $p_t < \bar{d}$ , since this means that the asset looks 'cheap' and will sell the asset if  $p_t > \bar{d}$ , since this means that the asset looks 'dear'. Assuming that value-based traders' demand is driven by the divergence of the price from the intrinsic value is a simple way of representing trading motivated by immediate profit opportunities.

We define  $X_t$  as the aggregate flow of trades coming from strategic traders in period  $t$ . The equilibrium market price of the asset is determined by the market clearing condition  $X_t + y_t = 0$  and the resulting price is given as

$$p_t = \bar{d} + \lambda X_t. \quad (2)$$

Equation 2 describes how the demand flow from arbitrageurs affects the market price of the asset. If the arbitrageur is buying the asset ( $X_t > 0$ ), the price,  $p_t$ , will be higher than it would be in the absence of an arbitrageur. The reverse will be true if the arbitrageur is selling the asset ( $X_t < 0$ ).

### 2.2.2 Strategic traders

The other set of agents in our model will attempt to profit from price deviations; we refer to them as strategic traders or arbitrageurs. Trading by strategic traders affects the market price of the risky asset.

We model strategic traders to be risk-averse with CARA-coefficient  $\alpha$  who maximize their expected final period utility, that is

$$\max -E \left[ \exp \left( -\alpha W_1^i \right) \right].$$

We define the wealth (or capital) of strategic trader  $i$ ,  $W_1^i$ , at period 1 as

$$W_1^i = M_1^i + e_1^i d,$$

where  $e_t^i$  is her position in the risky asset and  $M_t^i$  is her position in the riskless asset after trading in period  $t$ . The dynamics of  $e_t^i$  and  $M_t^i$  are simply given by the following equations:

$$e_t^i = e_{t-1}^i + x_t^i,$$

where  $e_{t-1}^i$  is the after-trade position at the end of period  $(t-1)$  and  $x_t^i$  is the trade order in period  $t$ ; and

$$M_t^i = M_{t-1}^i - p_t x_t^i,$$

which means that the strategic trader's investment in the risk-free asset changes only by the payments for purchases/sales of the risky asset. Negative values of  $M_t^i$  represent amounts borrowed, negative values of  $e_t^i$  mean being short. For simplicity we denote their starting positions as  $M^i \equiv M_{-1}^i$  and  $e^i \equiv e_{-1}^i$ .

Agents have CARA-utility on their normally distributed final wealth, hence their optimization problem is equivalent to:

$$\max_{x_0^i, x_1^i} CE(W_1^i) = M^i - \sum_{t=0}^1 p_t (x_t^i) x_t^i + \bar{d} \left( e^i + \sum_{t=0}^1 x_t^i \right) - \frac{\alpha}{2} \sigma^2 \left( e^i + \sum_{t=0}^1 x_t^i \right)^2. \quad (3)$$

In each period: (i) a determination of the strategic trader's solvency is made – if the constraint is not met, i.e. she goes default, the arbitrageur is forced to liquidate all positions and exit the market; if the constraint is met, i.e. she is solvent, the arbitrageur can continue to trade; (ii) the risky asset is traded. We discuss the constraints faced by strategic traders in the next subsection.

## 2.3 Constraints

Arbitrageurs in financial markets are often required to back their trading positions with their own capital. Examples include banks subject to minimum capital requirements and margin requirements in futures contracts. We will use a simple version of a leverage constraint in our analysis, but the model can accommodate different types of financial constraints, as long as these are related to market prices. We define the constraint in terms of the value of the arbitrageurs' portfolio.

Using our previous notation, we require the existing capital of a strategic trader prior to trading at date  $t$  ( $t = 0$  or  $1$ ) to be nonnegative, i.e.

$$W_{t-1}^i = M_{t-1}^i + p_{t-1}e_{t-1}^i \geq 0,$$

where the portfolio wealth consists of the riskless position of the trader and her risky endowment evaluated at the current market price. For simplicity we assume that agents are all solvent at the beginning of the model, evaluated at the unconditional mean price  $p_{-1} \equiv \bar{d}$ , i.e.  $W^i = M^i + p_{-1}e^i = M^i + \bar{d}e^i \geq 0$ . For the determination of trader  $i$ 's solvency prior to the date 1 trade, the constraint to be met is

$$W_0^i = M_0^i + p_0e_0^i > 0.$$

Using the dynamics of the risky and the riskless positions this constraint can be written in the following form:

$$W_0^i = M_0^i + p_0e_0^i = (M^i - p_0x_0^i) + p_0(e^i + x_0^i) = M^i + p_0e^i \geq 0. \quad (4)$$

This is useful because  $M^i$  and  $e^i$  are known before date 0 and hence it is only the  $p_0$  term that depends on the trading activity of the agents in period 0. From now on we will use this latest inequality to determine the solvency of strategic traders.

## 3 Trading strategy of a constrained monopoly

The optimization problem of the constrained monopolist arbitrageur can be written as before with the extra requirement for liquidation if being insolvent:

$$\max_{x_0, x_1} CE(W_1) = M - \sum_{t=0}^1 p_t(x_t)x_t + \bar{d} \left( e + \sum_{t=0}^1 x_t \right) - \frac{\alpha}{2} \sigma^2 \left( e + \sum_{t=0}^1 x_t \right)^2 \quad (5)$$

$$\begin{aligned}
\text{s.t. market clears} & : p_t = \bar{d} + \lambda x_t; \\
\text{dynamic budget constraints} & : M_t = M_{t-1} - p_t x_t \\
& \text{and } e_t = e_{t-1} + x_t; \\
\text{final payoff} & : W_1 = M_1 + de_1; \\
\text{insolvency constraint} & : x_1 = -e_0 \text{ if } W_0 = m_0 + p_0 e_0 \leq 0.
\end{aligned}$$

**Proposition 1** *There exist thresholds  $\bar{k}^m$  and  $\underline{k}^m$  such that we have the following equilibria:*

*if  $W = M + \bar{d}e \geq \lambda \bar{k}^m e^2$  (where  $0 \leq \bar{k}^m \leq 1$ ), the first best is feasible and the agent's optimal trade order is the same as in the absence of the constraint:  $x_0^u = x_1^u = -\frac{1}{2} \frac{\alpha \sigma^2}{\lambda + \alpha \sigma^2} e$  while  $p_0^u = p_1^u = \bar{d} - \frac{\lambda}{2} \frac{\alpha \sigma^2}{\lambda + \alpha \sigma^2} e$ ;*

*if  $\lambda \bar{k}^m e^2 > M + \bar{d}e \geq \lambda \max\{0, \underline{k}^m\} e^2$ , the agent reduces its date-0 sell order to stay solvent, that is  $x_0^c = -\frac{1}{\lambda} \frac{M + \bar{d}e}{e}$  and  $p_0^c = -\frac{M}{e}$ , then proceeds with  $x_1^c = -\frac{\alpha \sigma^2}{2\lambda + \alpha \sigma^2} e_0^c$  and  $p_1^c = \bar{d} - \lambda \frac{\alpha \sigma^2}{2\lambda + \alpha \sigma^2} e_0^c$ ;*

*if  $\underline{k}^m e^2 > M + \bar{d}e \geq 0$ , the trader liquidates; sells half of her endowment in both periods:  $x_0^l = x_1^l = -\frac{1}{2} e$  at prices  $p_0^l = p_1^l = \bar{d} - \frac{\lambda}{2} e$ ,*

*where  $\bar{k}^m$  and  $\underline{k}^m$  are functions of  $\lambda/\alpha \sigma^2$  and are given in Appendix ??, and  $\underline{k}^m \leq 0$  if and only if  $\lambda/\alpha \sigma^2 \geq 1/2$  - in this case the third situation never happens.*

The results of Proposition 1 are plotted on Figure 1. We find that if the proportion of wealth invested in the risky asset is low, i.e. trader is wealthy enough compared to her risky position, the solvency constraint will not bind. In this case she smoothes her trade orders across periods in order to minimize her price impact and hence trades the same amount in both periods.

As the proportion of wealth invested in the risky asset becomes higher, i.e. she has limited starting wealth relative to the risky part of the portfolio (i.e.  $\lambda \bar{k}^m e^2 > M + \bar{d}e$  for some  $\bar{k}^m$ ), the agent faces a trade-off between the optimal risk-sharing and her price impact. If the market is relatively illiquid or the agent has low risk-aversion parameter or the asset is not very risky, i.e.  $\lambda/\alpha \sigma^2 \geq 1/2$ , she does not want to become insolvent and bear the high cost of the fire-sale, hence she reduces trade speed. The agent will also trade less in the first period if the market is relatively liquid ( $\lambda/\alpha \sigma^2 < 1/2$ ) and her initial wealth is high enough ( $M + \bar{d}e \geq \lambda \bar{k}^m e^2$ ).

Finally, if the market is relatively liquid compared to the trader's risk-bearing capacity and much of her portfolio wealth is invested in the risky asset, she does not mind violating the solvency constraint and the liquidation. In this case she smoothes her trade orders across periods in order to minimize her price impact and hence trades away the same amount in both periods, i.e. half of her initial endowment.

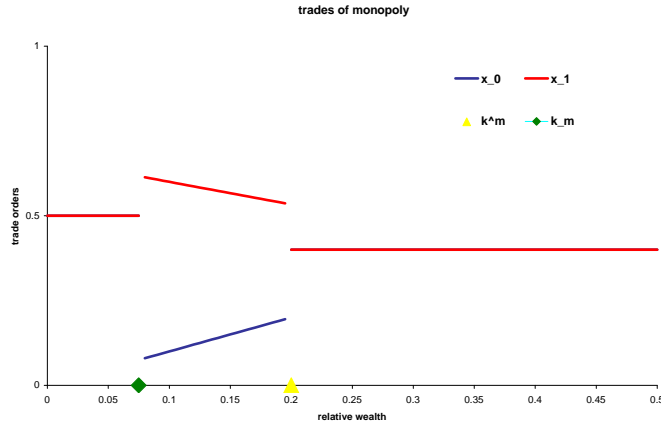


Figure 1: Optimal trade of the constrained monopoly as a function of the inverse of the proportion of total wealth invested in the risky asset.

## 4 Duopoly with same positions in both assets

Before proceeding to the general model of two strategic traders, we examine the case when these agents are identical, that is they start with positions  $M^1 = M^2 = M$  and  $e^1 = e^2 = e \neq 0$ . The optimization problem of strategic trader  $i$  is:

$$\max_{x_0^i, x_1^i} CE(W_1^i) = M - \sum_{t=0}^1 p_t (x_t^i) x_t^i + \bar{d} \left( e + \sum_{t=0}^1 x_t^i \right) - \frac{\alpha}{2} \sigma^2 \left( e + \sum_{t=0}^1 x_t^i \right)^2 \quad (6)$$

$$\text{s.t. market clears} : p_t = \bar{d} + \lambda \left( \sum_{i=1}^2 x_t^i \right);$$

$$\text{dynamic budget constraints} : M_t^i = M_{t-1}^i - p_t x_t^i$$

$$\text{and } e_t^i = e_{t-1}^i + x_t^i;$$

$$\text{final payoff} : W_1^i = M_1^i + de_1^i;$$

$$\text{insolvency constraint} : x_1^i = -e_0^i \text{ if } W_0^i = M_0^i + p_0 e_0^i < 0;$$

We ignore the case when  $e = 0$  because a risk-averse agent not owning the risky asset will not trade at all with the value-traders as she would have to offer a price  $p_t > \bar{d}$  if she wanted to buy, and accept a price  $p_t < \bar{d}$  at which she could go short. These restrictions on the starting positions also yield that  $W_0^1 = M_0^1 + p_0 e_0^1 = M^1 + p_0 e^1 = M^2 + p_0 e^2 = M_0^2 + p_0 e_0^2 = W_0^2$ , hence the insolvency constraint binds for both agents at the same time. Therefore in equilibrium agents need to have the same certainty equivalent and they must pursue identical trading strategies otherwise one of them would deviate.



**Proposition 2** *There exist thresholds  $\bar{k}^d$  and  $\underline{k}^d$  such that we have the following equilibria:*

*if  $W = M + \bar{d}e \geq \lambda \bar{k}^d e^2$  (where  $0 \leq \bar{k}^d \leq 1$ ), the first best is feasible and the agents' optimal trade orders are the same as in the absence of the constraint:  $x_0^{1u} = x_0^{2u} = -a_0^{ss}e$  and  $x_1^s = x_1^w = -a_1^{ss}e_0$  with  $p_0^u = \bar{d} - 2\lambda a_0^{ss}e$  and  $p_1^u = \bar{d} + \lambda(2b_1^{ss} - a_1^{ss})e_0$ ;*

*if  $\lambda \bar{k}^d e^2 > M + \bar{d}e \geq \lambda \max\{0, \underline{k}^d\} e^2$ , both agents reduce their date-0 sell orders to stay solvent, that is  $x_0^{1c} = x_0^{2c} = -\frac{1}{2\lambda} \frac{M + \bar{d}e}{e}$  and  $p_0^c = -\frac{M}{e}$ , then proceed with  $x_1^{1c} = x_1^{2c} = -a_1^{ss}e_0^c$  and  $p_1^c = \bar{d} - 2\lambda a_1^{ss}e_0^c$ ;*

*if  $\lambda \underline{k}^d e^2 > M + \bar{d}e \geq 0$ , the traders liquidate; sell half of their endowment in both periods:  $x_0^{1l} = x_0^{2l} = x_1^{1l} = x_1^{2l} = -\frac{1}{2}e$  at prices  $p_0^l = p_1^l = \bar{d} - \lambda e$ ,*

*where  $\bar{k}^d$  and  $\underline{k}^d$  are functions of  $\lambda/\alpha\sigma^2$  given in Appendix ?? and  $\underline{k}^d \leq 0$  if and only if  $\lambda/\alpha\sigma^2 \geq \bar{l}^d$  constant - in this case the third situation never happens.*

The results of Proposition 2 are plotted on Figure 2. The intuition is similar to that of the monopoly: We find that as long as agents have a low proportion of wealth invested in the risky asset, the solvency constraint will not bind. In this case they smooth their trade orders across periods in order to minimize their price impact and hence trade the same amount in both periods.

As the proportion of wealth invested in the risky asset becomes higher, i.e.  $\lambda \bar{k}^d e^2 \leq M + \bar{d}e$ , agents face a trade-off between the optimal risk-sharing and the trading speed. If the market is very illiquid or the agents have low risk-aversion parameters or the asset is not very risky, that is  $\lambda/\alpha\sigma^2 \geq \bar{l}^d$  for a given constant  $\bar{l}^d$ , they do not want to become insolvent and hence reduce trade speed. They also want to avoid insolvency and fire-sale, and hence reduce their trade speed if the market is relatively liquid or they are significantly risk-averse ( $\lambda/\alpha\sigma^2$  is close to zero), and they are relatively not poor, i.e.  $M + \bar{d}e \geq \lambda \underline{k}^d e^2$ .

Finally if the market is relatively liquid compared to the traders' risk-bearing capacity or the payoff risk is high ( $\lambda/\alpha\sigma^2 < \bar{l}^d$ ) and most of their capital is invested in the risky asset, they decide to violate the constraint and hence liquidate. In this case they smooth their trade orders across periods in order to minimize their price impact and hence sell the same amounts in both periods: half of their initial endowments.

The only changes compared to the monopolistic model are related to the thresholds, and it is because of the fact that agents are identical. When one decides on a particular trade order, she has to take into account that altogether they will have a price impact double of that in the single agent case. Comparing trade orders, prices, and the wealth thresholds (the  $\underline{k}$ s and the  $\bar{k}$ s) in the monopoly and the duopoly case we find that the unconstrained trade order is less per se and the equilibrium price is lower; the constrained price is the same as in the monopolistic case and trades are half of the original; while

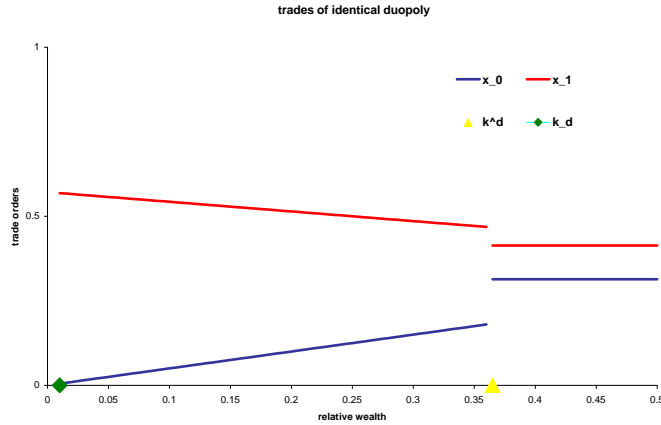


Figure 2: Optimal trade of the identical duopoly as a function of the inverse of the proportion of total wealth invested in the risky asset.

when being liquidated, trades are the same in both cases and the market-clearing price in case of multiple agents is lower.

## 5 Duopoly with different cash but same risky positions

In this section we study the optimal trading strategies of two strategic agents who start with the same amount in the risky asset but are heterogenous in the riskless asset position. As described in Section 2.3, the solvency constraint can be rewritten as

$$M^i + p_0 e \geq 0$$

for  $i = 1, 2$ , hence by making  $M^1$  and  $M^2$  different we can ensure that the constraint will not bind for both agents at the same time and therefore it is possible to obtain an equilibrium price  $p_0$  under which one agent remains solvent while the other is forced to go for a fire-sale in the subsequent period. If, for example,  $M^1 > M^2$ , that is agent one is wealthier than agent two, the solvency of agent two ( $W_0^2 = M^2 + p_0 e \geq 0$ ) will also imply the solvency of agent one, as  $W_0^1 = M^1 + p_0 e > M^2 + p_0 e = W_0^2 \geq 0$ . Therefore, from now on we will call the two agents strong and weak, where  $M^s > M^w$ .

The optimization problem of agent  $i$  is the following:

$$\max_{x_0^i, x_1^i} CE(W_1^i) = M^i - \sum_{t=0}^1 p_t(x_t^i) x_t^i + \bar{d} \left( e + \sum_{t=0}^1 x_t^i \right) - \frac{\alpha}{2} \sigma^2 \left( e + \sum_{t=0}^1 x_t^i \right)^2 \quad (7)$$

$$\begin{aligned}
\text{s.t. market clears} & : p_t = \bar{d} + \lambda \left( \sum_{i=1}^2 x_t^i \right); \\
\text{dynamic budget constraints} & : M_t^i = M_{t-1}^i - p_t x_t^i \\
& \text{and } e_t^i = e_{t-1}^i + x_t^i; \\
\text{final payoff} & : W_1^i = M_1^i + d e_1^i; \\
\text{and insolvency constrain} & : x_1^i = -e_0^i \text{ if } W_0^i = M^i + p_0 e < 0.
\end{aligned}$$

For the definition of the equilibrium we first define the value function.

**Definition 3** We define the following 'conditional value functions' for period  $t$ :

$$V_t^{jk} (M_t^i, e_t^i, M_t^{-i}, e_t^{-i}) = M_t^i + \bar{d} e_t^i - \frac{1}{2} (e_t^i, e_t^{-i}) Q_t^{jk} (e_t^i, e_t^{-i})'$$

for  $i \in \{s, w\}$  and  $t = 0, 1$ , conditional on state  $\{jk\}$ , where

$j \in \{s, l\}$  denotes the state of agent  $i$ , that is if she is solvent or being liquidated;

$k \in \{s, l\}$  denotes the state of agent  $-i$ ;

$M_t^i$  and  $e_t^i$  are the after-trade portfolio holdings of agent  $i$ ;

$M_t^{-i}$  and  $e_t^{-i}$  are the after-trade portfolio holdings of agent  $-i$ ;

$Q_t^{jk}$  is a  $2 \times 2$  symmetric matrix.

**Definition 4** The value function of agent  $i$  at date  $t$  is

$$V_t^i (M_t^i, e_t^i, M_t^{-i}, e_t^{-i}) = \begin{cases} V_t^{ss} (M_t^i, e_t^i, M_t^{-i}, e_t^{-i}) & \text{if } W_t^i, W_t^{-i} \geq 0, \\ V_t^{sl} (M_t^i, e_t^i, M_t^{-i}, e_t^{-i}) & \text{if } W_t^i \geq 0 > W_t^{-i}, \\ V_t^{ls} (M_t^i, e_t^i, M_t^{-i}, e_t^{-i}) & \text{if } W_t^{-i} \geq 0 > W_t^i, \\ V_t^{ll} (M_t^i, e_t^i, M_t^{-i}, e_t^{-i}) & \text{if } 0 \geq W_t^i, W_t^{-i}. \end{cases}$$

**Definition 5** A Nash-equilibrium of the above trading game is a vector of demands  $\{x_t^i\}_{i=s,w;t=0,1}$  such that  $x_1^i$  solves the program

$$\begin{aligned}
& \max_x V_1^i (M_1^i, e_1^i, M_1^{-i}, e_1^{-i} | x_1^{-i}, M_0^i, e_0^i, M_0^{-i}, e_0^{-i}) \\
& = \max_x V_1^i (M_0^i - P(x)x (P(x)), e_0^i + x (P(x)), M_0^{-i} - P(x)x_1^{-i}, e_1^{-i})
\end{aligned}$$

and  $x_0^i$  solves the program

$$\begin{aligned}
& \max_x V_0^i (M_0^i, e_0^i, M_0^{-i}, e_0^{-i} | x_0^{-i}, M^i, M^{-i}, e) \\
& = \max_x V_0^i (M^i - P(x)x (P(x)), e + x (P(x)), M^{-i} - P(x)x_0^{-i}, e_0^{-i})
\end{aligned}$$

with

$$V_0^i (\cdot) = \max \{V_0^{ss} (\cdot), V_0^{sl} (\cdot), V_0^{ls} (\cdot), V_0^{ll} (\cdot)\},$$

and  $P_{(x)}$  is the market-clearing price in period  $t$  when agent  $i$  submits the demand  $x$ , and agent  $-i$  submits her equilibrium demand  $x_t^{-i}$  and  $p_t$  clears the market at date  $t$ .

The above problem can be solved backwards: first we solve for the optimal trades at date 1 given the state we are in (that is whether the traders are solvent or insolvent), then obtain value functions representing the continuation utilities, and finally we solve for the optimal trades of period 0.

## 5.1 Optimal trades in period 1

Based on the solvency constraint we have to distinguish three states of the world at date 1: either both agents are solvent, that is  $W_0^s, W_0^w \geq 0$ ; the strong agent is solvent while the weak is insolvent, that is  $W_0^s \geq 0 > W_0^w$ ; or both agents are insolvent, which is equivalent to  $0 > W_0^s, W_0^w$ .

### 5.1.1 Both agents are solvent

When  $W_0^s, W_0^w \geq 0$ , for which a sufficient condition is  $M^w + p_0 e \geq 0$ , both agents remain solvent and hence they do not face any constraints regarding their period 1 trades. Given the before-trade positions  $\{M_0^i, e_0^i\}_{i=s,w}$ , agent  $i$ 's optimization problem ( $i = s, w$ ) is

$$\begin{aligned} \max_{x_1^i} CE(W_1^i) &= M_0^i - p_1 (x_1^i) x_1^i + \bar{d} (e_0^i + x_1^i) - \frac{\alpha}{2} \sigma^2 (e_0^i + x_1^i)^2 \\ \text{s.t. market clears: } &p_1 = \bar{d} + \lambda (x_1^i + x_1^{-i}) \end{aligned} \quad (8)$$

We have the following result:

**Proposition 6** *If both agents are solvent, the optimal period 1 trades are given by*

$$x_1^{i*} = -a_1^{ss} e_0^i + b_1^{ss} e_0 = -\frac{\alpha \sigma^2}{\lambda + \alpha \sigma^2} e_0^i + \frac{\alpha \sigma^2}{\lambda + \alpha \sigma^2} \frac{\lambda}{3\lambda + \alpha \sigma^2} e_0$$

and the market-clearing price is

$$p_1^* = \bar{d} - \lambda \frac{\alpha \sigma^2}{3\lambda + \alpha \sigma^2} e_0$$

where  $e_0 = e_0^s + e_0^w$ .

**Proof.** See Appendix C.1. ■

The optimal trade orders are linear functions of stock holdings before trade and the average stock holdings of the two strategic traders. For the coefficient of the endowment

of agent  $i$ ,  $0 \leq a_1^{ss} \leq 1$ , trade thus reduces the dispersion in stock holdings. The net order of strategic agents,  $x_1^{s*} + x_1^{w*} = -\alpha\sigma^2 / (3\lambda + \alpha\sigma^2) e_0$ , is negative, as risk-averse strategic traders prefer to sell part of their holdings at a discount to the long-term traders for risk-sharing purposes. The coefficient  $\alpha\sigma^2 / (3\lambda + \alpha\sigma^2)$  measures the speed at which disperse stock holdings become identical over time and risk is optimally shared.

The above result also yields that when both agents remain solvent and reach date 1 with positions  $\{M_0^i, e_0^i\}_{i=s,w}$ , the value function becomes

$$\begin{aligned} V_0^{ss}(M_0^i, e_0^i, M_0^{-i}, e_0^{-i}) &\equiv M_0^i - p_1^*(x_1^{i*}) x_1^{i*} + \bar{d}(e_0^i + x_1^{i*}) - \frac{\alpha}{2}\sigma^2 (e_0^i + x_1^{i*})^2 \\ &= M_0^i + \bar{d}e_0^i - \frac{1}{2}\alpha\sigma^2 (e_0^i)^2 + \frac{1}{2}(e_0^i, e_0^{-i})' Q_0^{ss} (e_0^i, e_0^{-i})'. \end{aligned}$$

where  $Q_0^{ss}$  is a  $2 \times 2$  positive definite matrix:

$$Q_0^{ss} = \frac{1}{2\lambda + \alpha\sigma^2} \left( \frac{\alpha\sigma^2}{\lambda + \alpha\sigma^2} \right)^2 \left( \frac{2\lambda + \alpha\sigma^2}{3\lambda + \alpha\sigma^2} \right)^2 \begin{pmatrix} (2\lambda + \alpha\sigma^2)^2 & -\lambda(2\lambda + \alpha\sigma^2) \\ -\lambda(2\lambda + \alpha\sigma^2) & \lambda^2 \end{pmatrix}$$

### 5.1.2 One agent is solvent, the other liquidates

If the weak agent violates the first-period constraint while the strong agent remains solvent, that is  $W_0^s \geq 0 > W_0^w$ , the weak agent will have to liquidate everything, therefore

$$x_1^w = -e_0^w$$

while the strong agent faces the usual optimization problem, that is

$$\max_{x_1^s} CE(W_1^s) = M_0^s - p_1(x_1^s) x_1^s + \bar{d}(e_0^s + x_1^s) - \frac{\alpha}{2}\sigma^2 (e_0^s + x_1^s)^2 \quad (9)$$

$$\text{s.t. market clears: } p_1 = \bar{d} + \lambda(x_1^s + x_1^w).$$

**Proposition 7** *If the strong agent remains solvent while the weak agent is forced to liquidate in period 1, the optimal trades are*

$$x_1^{w*} = -e_0^w$$

$$x_1^{s*} = -\frac{\alpha\sigma^2}{2\lambda + \alpha\sigma^2} e_0^s + \frac{\lambda}{2\lambda + \alpha\sigma^2} e_0^w$$

and the market-clearing price is

$$p_1^{sl*} = \bar{d} - \lambda \frac{\alpha\sigma^2}{2\lambda + \alpha\sigma^2} e_0^s - \lambda \frac{\lambda + \alpha\sigma^2}{2\lambda + \alpha\sigma^2} e_0^w.$$

**Proof.** See Appendix C.2. ■

The optimal trade of the strong agent is linear combination of her before-trade holdings and the amount is sold by the distressed trader: she sells some of her endowment to share risk with the long-term traders and also buys at a discount from the liquidated trader. The sign of the net trade depends on the holdings of the two agents and the relative liquidity of the market to the risk-bearing capacity of the traders: lower  $\lambda/\alpha\sigma^2$  implies lower  $\lambda/(2\lambda + \alpha\sigma^2)$  and higher  $\alpha\sigma^2/(2\lambda + \alpha\sigma^2)$ , hence when the market is relatively deep, the risk-sharing motive from own holdings dominates the price-impact of purchasing the liquidated assets of the distressed trader.

The above result also yields that when the strong agent remains solvent with positions  $\{M_0^s, e_0^s\}$  and the weak is insolvent with positions  $\{M_0^w, e_0^w\}$ , the value functions become

$$\begin{aligned} V_0^{sl}(M_0^s, e_0^s, M_0^l, e_0^l) &= M_0^s - p_1^*(x_1^{s*})x_1^{s*} + \bar{d}(e_0^s + x_1^{s*}) - \frac{\alpha}{2}\sigma^2(e_0^s + x_1^{s*})^2 \\ &= M_0^s + \bar{d}e_0^s - \frac{\alpha}{2}\sigma^2(e_0^s)^2 + \frac{1}{2}(e_0^s, e_0^w)Q_0^{sl}(e_0^s, e_0^w)', \end{aligned}$$

where  $Q_0^{sl}$  is a  $2 \times 2$  positive semidefinite matrix:

$$Q_0^{sl} = \frac{1}{2\lambda + \alpha\sigma^2} \begin{pmatrix} (\alpha\sigma^2)^2 & -\lambda\alpha\sigma^2 \\ -\lambda\alpha\sigma^2 & \lambda^2 \end{pmatrix}$$

and

$$\begin{aligned} V_0^{ls}(M_0^w, e_0^w, M_0^s, e_0^s) &= M_0^w - p_1^*(x_1^{w*})x_1^{w*} + \bar{d}(e_0^w + x_1^{w*}) - \frac{\alpha}{2}\sigma^2(e_0^w + x_1^{w*})^2 \\ &= M_0^w + \bar{d}e_0^w - \frac{1}{2}(e_0^w, e_0^s)Q_0^{ls}(e_0^w, e_0^s)', \end{aligned}$$

where  $Q_0^{ls}$  is a  $2 \times 2$  matrix:

$$Q_0^{ls} = \frac{\lambda}{2\lambda + \alpha\sigma^2} \begin{pmatrix} 2(\lambda + \alpha\sigma^2) & \alpha\sigma^2 \\ \alpha\sigma^2 & 0 \end{pmatrix}.$$

### 5.1.3 Both agents liquidate

If both agents violate the first-period constraint, that is  $0 > W_0^s, W_0^w$ , they will have to liquidate everything at date 1, therefore

$$x_1^i = -e_0^i$$

for  $i = s, w$ . It means that the market-clearing price is

$$p_1 = \bar{d} - \lambda(e_0^s + e_0^w).$$

The above result also yields that when both agents get liquidated with positions  $\{M_0^s, e_0^s\}$  and  $\{M_0^w, e_0^w\}$ , their value functions become

$$\begin{aligned} V_0^u(M_0^i, e_0^i, M_0^{-i}, e_0^{-i}) &= M_0^i - p_1^*(x_1^{i*})x_1^{i*} + \bar{d}(e_0^i + x_1^{i*}) - \frac{\alpha}{2}\sigma^2(e_0^i + x_1^{i*})^2 \\ &= M_0^i + \bar{d}e_0^i - \frac{1}{2}(e_0^i, e_0^{-i})Q_0^u(e_0^i, e_0^{-i})', \end{aligned}$$

where

$$Q_0^u = \begin{pmatrix} 2\lambda & \lambda \\ \lambda & 0 \end{pmatrix}.$$

## 5.2 Optimal trades in period 0

In this subsection we examine the date 0 trades in equilibrium. When determining the optimal trades we conjecture a state of the world, then we solve for the optimal period 0 trades given we are in the conjectured state (which will satisfy that noone is willing to deviate from it as long as the state of the world is unchanged), and finally we check whether it is optimal to deviate for any agent in such a way that would change the state of the world as in those scenarios the change of the state would imply a change in the value function as well.

Based on the solvency constraint we have to distinguish three states of the world at date 1: either both agents are solvent, that is  $W_0^s, W_0^w \geq 0$ ; the strong agent is solvent while the weak is insolvent, that is  $W_0^s \geq 0 > W_0^w$ ; or both agents are insolvent, which is equivalent to  $0 > W_0^s, W_0^w$ .

### 5.2.1 Equilibrium of type 1: Both agents remain solvent

If both agents remain solvent for period 1, their optimal trades are

$$x_1^i = -a_1^{ss}e_0^i + b_1^{ss}e_0 = -\frac{\alpha\sigma^2}{\lambda + \alpha\sigma^2}e_0^i + \frac{\alpha\sigma^2}{\lambda + \alpha\sigma^2}\frac{\lambda}{3\lambda + \alpha\sigma^2}e_0,$$

with the market-clearing price

$$p_1 = \bar{d} - \lambda\frac{\alpha\sigma^2}{3\lambda + \alpha\sigma^2}e_0,$$

while their value functions become

$$\begin{aligned} V_0^{ss} (M_0^i, M_0^{-i}, e_0^i, e_0^{-i}) &\equiv \max_{x_1^i} M_0^i - p_1 (x_1^i) x_1^i + \bar{d} (e_0^i + x_1^i) - \frac{\alpha}{2} \sigma^2 (e_0^i + x_1^i)^2 \\ &= M_0^i + \bar{d} e_0^i - \frac{\alpha}{2} \sigma^2 (e_0^i)^2 + \frac{1}{2} (e_0^i, e_0^{-i}) Q_0^{ss} (e_0^i, e_0^{-i})' \end{aligned}$$

where  $Q_0^{ss}$  is a  $2 \times 2$  symmetric, positive definite matrix:

$$Q_0^{ss} = \frac{1}{2\lambda + \alpha\sigma^2} \left( \frac{\alpha\sigma^2}{\lambda + \alpha\sigma^2} \right)^2 \left( \frac{2\lambda + \alpha\sigma^2}{3\lambda + \alpha\sigma^2} \right)^2 \begin{pmatrix} (2\lambda + \alpha\sigma^2)^2 & -\lambda(2\lambda + \alpha\sigma^2) \\ -\lambda(2\lambda + \alpha\sigma^2) & \lambda^2 \end{pmatrix}.$$

Therefore the optimal first-period trades satisfy the following optimization problems:

$$\begin{aligned} \max_{x_0^i} V_0^{ss} (M_0^i, M_0^{-i}, e_0^i, e_0^{-i}) &= M_0^i + \bar{d} e_0^i - \frac{\alpha}{2} \sigma^2 (e_0^i)^2 + \frac{1}{2} (e_0^i, e_0^{-i}) Q_0^{ss} (e_0^i, e_0^{-i})' \\ &= M^i + \bar{d} e + (\bar{d} - p_0) x_0^i - \frac{\alpha}{2} \sigma^2 (e + x_0^i)^2 \\ &\quad + \frac{1}{2} (e + x_0^s, e + x_0^w) Q_0^{ss} (e + x_0^s, e + x_0^w)'. \end{aligned}$$

After solving for the optimal trades conditional on both guys satisfying the solvency constraint, we obtain the following result:

**Proposition 8** *There exists an equilibrium in which both agents remain solvent with trades and market-clearing prices*

$$\begin{aligned} x_0^i &= -a_0^{ss} e \text{ and } x_1^i = -a_1^{ss} e_0^i + b_1^{ss} e_0 \text{ for } i = s, w \text{ with prices} \\ p_0 &= \bar{d} - 2\lambda a_0^{ss} e \text{ and } p_1 = \bar{d} + \lambda (2b_1^{ss} - a_1^{ss}) e_0. \end{aligned}$$

with the necessary and sufficient condition

$$0 \leq \lambda \max \{ \underline{k}^{ss,sl}, 2a_0^{ss} \} e^2 \leq M^w + \bar{d}e \leq M^s + \bar{d}e,$$

where the function  $\underline{k}^{ss,sl}(\lambda/\alpha\sigma^2)$  is given in Appendix D.1.1.

Given the state of the world in which both agents remain solvent, the above trades and prices must be consistent with satisfying the solvency constraint, that is

$$M^s + p_0 e \geq M^w + p_0 e \geq 0,$$

which is equivalent to

$$M^s + \bar{d}e \geq M^w + \bar{d}e \geq 2\lambda a_0^{ss} e^2.$$



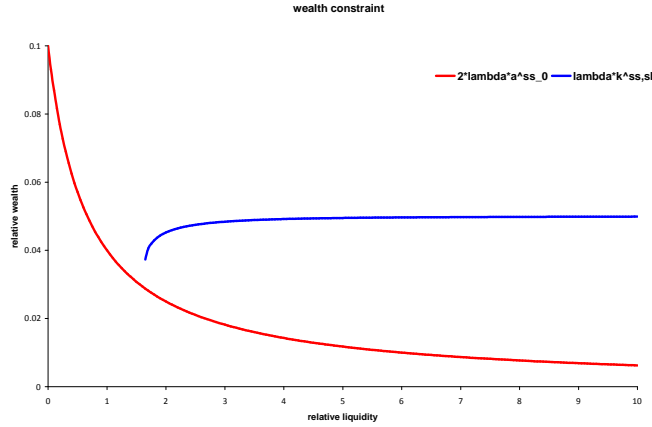


Figure 3: Two constraints for portfolio wealth when both agent remain solvent.

For the existence of a Nash equilibrium, though, we need another constraint on the starting wealth:

$$\lambda \underline{k}^{ss,sl} e^2 \leq M^w + \bar{d}e.$$

where  $\underline{k}^{ss,sl}$  is a function of the relative market depth,  $\lambda/\alpha\sigma^2$ . The reason for this is rather simple. Being aware of the solvency constraint, agents can engage in the costly manipulation of date 0 prices if they are able to extract higher payoffs in the next trading round. In this scenario the strong agent might want to force the weak agent to liquidation, as it would mean a lower purchase price in the second round. The cost of making the weak agent distressed is increasing in her starting wealth,  $M^w + \bar{d}e$ , hence there exists a threshold  $\underline{k}^{ss,sl}$  such that the strong agent will engage in this type of price-manipulation if and only if  $M^w + \bar{d}e < \lambda \underline{k}^{ss,sl} e^2$ .

The function  $a_0^{ss} = 1/[3(\lambda/\alpha\sigma^2) + 2]$  is strictly decreasing in  $\lambda/\alpha\sigma^2$  with limits

$$\lim_{\lambda/\alpha\sigma^2 \rightarrow 0} a_0^{ss} = \frac{1}{2} \text{ and } \lim_{\lambda/\alpha\sigma^2 \rightarrow \infty} \frac{\alpha\sigma^2}{3\lambda + 2\alpha\sigma^2} = 0.$$

Here  $\underline{k}^{ss,sl}$  will only exist for  $\lambda/\alpha\sigma^2 \geq 2.85$  and it is a strictly increasing function and has limit

$$\lim_{\lambda/\alpha\sigma^2 \rightarrow \infty} \underline{k}^{ss,sl} = \frac{1}{2}.$$

These two requirements mean that we need

$$\lambda \max \{ \underline{k}^{ss,sl}, 2a_0^{ss} \} e^2 \leq M^w + \bar{d}e \leq M^s + \bar{d}e.$$

On Figure 3 we plot the LHS of this inequality. The thin line represents the price

constraint  $(2\lambda\frac{\alpha\sigma^2}{3\lambda+2\alpha\sigma^2})$  and the thick line stands for the deviation constraint  $(\lambda\underline{k}^{ss,sl})$ . Agents' starting portfolio wealth must be above both constraint. We see that in relatively liquid markets (where  $\lambda/\alpha\sigma^2$  is low), the price impact of traders is small hence price manipulation to the extent at which it would trigger the liquidation of the weak agent is costly. Therefore agents will not engage in this type of activity and therefore portfolio wealth only has to satisfy the price constraint, i.e., the constraint needed to have nonnegative intermediate wealth when executing the first-best trades.

### 5.2.2 Equilibrium of type 2: The strong agent is solvent, the weak agent is liquidated

If one agent is liquidated while the other survives with after-trade positions of  $e_1^s$  and  $e_1^w$  the second-period (optimal) trades are

$$x_1^w = -e_0^w$$

and

$$x_1^s = \frac{\lambda}{2\lambda + \alpha\sigma^2} e_0^w - \frac{\alpha\sigma^2}{2\lambda + \alpha\sigma^2} e_0^s$$

with market-clearing price

$$p_1 = \bar{d} - \lambda \frac{\lambda + \alpha\sigma^2}{2\lambda + \alpha\sigma^2} e_0^w - \lambda \frac{\alpha\sigma^2}{2\lambda + \alpha\sigma^2} e_0^s$$

and the continuation value functions are

$$V_0^{sl}(M_0^s, e_0^s, M_0^l, e_0^l) = M_0^s + \bar{d}e_0^s - \frac{\alpha}{2}\sigma^2 (e_0^s)^2 + \frac{1}{2}(e_0^s, e_0^w) Q_0^{sl} (e_0^s, e_0^w)',$$

where  $Q_0^{sl}$  is a  $2 \times 2$  positive semidefinite matrix:

$$Q_0^{sl} = \frac{1}{2\lambda + \alpha\sigma^2} \begin{pmatrix} (\alpha\sigma^2)^2 & -\lambda\alpha\sigma^2 \\ -\lambda\alpha\sigma^2 & \lambda^2 \end{pmatrix}$$

and

$$V_0^{ls}(M_0^w, e_0^w, M_0^s, e_0^s) = M_0^w + \bar{d}e_0^w - \frac{1}{2}(e_0^w, e_0^s) Q_0^{ls} (e_0^w, e_0^s)',$$

where  $Q_0^{ls}$  is

$$Q_0^{ls} = \frac{\lambda}{2\lambda + \alpha\sigma^2} \begin{pmatrix} 2(\lambda + \alpha\sigma^2) & \alpha\sigma^2 \\ \alpha\sigma^2 & 0 \end{pmatrix}.$$

Therefore the optimal first-period trades satisfy the following optimization problems:

$$\begin{aligned}\max_{x_0^s} V_0^{sl}(M_0^s, e_0^s, M_0^l, e_0^l) &= M_0^s + \bar{d}e_0^s - \frac{\alpha}{2}\sigma^2 (e_0^s)^2 + \frac{1}{2}(e_0^s, e_0^w) Q_0^{sl}(e_0^s, e_0^w)' \\ &= M^s + \bar{d}e + (\bar{d} - p_0)x_0^s - \frac{\alpha}{2}\sigma^2 (e + x_0^s)^2 \\ &\quad + \frac{1}{2}(e + x_0^s, e + x_0^w) Q_0^{sl}(e + x_0^s, e + x_0^w)'\end{aligned}$$

for the strong agent, while for the weak agent we have

$$\begin{aligned}\max_{x_0^w} V_0^{ls}(M_0^w, e_0^w, M_0^s, e_0^s) &= M_0^w + \bar{d}e_0^w - \frac{1}{2}(e_0^w, e_0^s) Q_0^{ls}(e_0^w, e_0^s)' \\ &= M^w + \bar{d}e + (\bar{d} - p_0)x_0^w - \frac{1}{2}(e_0^w, e_0^s) Q_0^{ls}(e_0^w, e_0^s)'\end{aligned}$$

After solving for the optimal trades conditional on the strong guy surviving and the weak trader being liquidated, we obtain the following result:

**Proposition 9** *There exists an equilibrium in which the strong trader remains solvent and the weak trader is liquidated with trades and market-clearing prices in the following form:*

$$\begin{aligned}x_0^s &= -a_0^{sl}e \text{ and } x_0^w = -a_0^{ls}e \text{ and } p_0 = \bar{d} - \lambda(a_0^{sl} + a_0^{ls})e \text{ and} \\ x_1^s &= \left[ \frac{\lambda}{2\lambda + \alpha\sigma^2} (1 - a_0^{ls}) - \frac{\alpha\sigma^2}{2\lambda + \alpha\sigma^2} (1 - a_0^{sl}) \right] e \text{ and } x_1^w = -(1 - a_0^{ls})e \text{ and} \\ p_1 &= \bar{d} - \lambda \left[ \frac{\lambda + \alpha\sigma^2}{2\lambda + \alpha\sigma^2} (1 - a_0^{ls}) + \frac{\alpha\sigma^2}{2\lambda + \alpha\sigma^2} (1 - a_0^{sl}) \right] e\end{aligned}$$

with the constants  $a^{sl}$  and  $a^{ls}$  (as functions of  $\lambda/\alpha\sigma^2$ ) given in Appendix D.2. A necessary condition for this type of equilibrium to happen is the existence of function  $\bar{k}^{ls,ss}(\lambda/\alpha\sigma^2)$  (given in Appendix D.2.1) such that

$$0 < M^w + \bar{d}e \leq \lambda \bar{k}^{ls,ss} e^2 < \lambda(a_0^{sl} + a_0^{ls})e^2 \leq M^s + \bar{d}e.$$

Given the state of the world in which the strong agent remains solvent and the other is liquidated, the above trades and prices must be consistent with only the weak trader violating the solvency constraint, that is

$$M^s + p_0e \geq 0 > M^w + p_0e,$$

which is equivalent to

$$M^s + \bar{d}e \geq \lambda(a_0^{sl} + a_0^{ls})e^2 > M^w + \bar{d}e.$$

For the existence of a Nash equilibrium, though, we need another constraints on the starting wealth:

$$\lambda \bar{k}^{ls,ss} e^2 > M^w + \bar{d}e,$$

where  $\bar{k}^{ls,ss}$  is a functions of the relative market depth,  $\lambda/\alpha\sigma^2$ . The reason for this is rather simple. Being aware of the solvency constraint, agents can engage in the costly manipulation of date 0 prices if they are able to extract higher payoffs in the next trading round. In this equilibrium the strong agent remains solvent while the weak is forced liquidation, therefore there is no possible benefit for the strong agent: either she could push herself to liquidation by bearing extra cost in period 0 and then facing a reduced action space (the forced fire-sale) or, again by bearing a cost, make sure that the other agent survives, which makes the period 1 risk-sharing more costly as the weak guy would trade in the same direction.

The weak agent, however, might benefit from price manipulation and "rescue" itself: in case she reduces her selling speed in period 0 to increase the price, she remains unconstrained and hence is able to trade optimally for risk-sharing. The cost of this activity is decreasing in her own wealth,  $M^w + \bar{d}e$ , hence there exists a threshold  $\bar{k}^{ls,ss}$  such that the weak agent will engage in this type of price-manipulation if  $M^w + \bar{d}e \geq \lambda \bar{k}^{ls,ss} e^2$ .

The above constraints yield that we have a price constraint for the strong agent; and a price and a deviation constraint for the weak agent: for the existence of an equilibrium we need the portfolio wealth of the strong trade to satisfy

$$M^s + \bar{d}e \geq \lambda (a_0^{sl} + a_0^{ls}) e^2,$$

while for the weak agent we must have

$$\lambda \min \left\{ \bar{k}^{ls,ss}, a_0^{sl} + a_0^{ls} \right\} e^2 > M^w + \bar{d}e \geq 0.$$

It is easy to show that  $\lambda (a_0^{sl} + a_0^{ls}) > \bar{k}^{ls,ss} > 0$ , hence the above constraints can be summed up as

$$0 \leq M^w + \bar{d}e \leq \lambda \bar{k}^{ls,ss} e^2 < \lambda (a_0^{sl} + a_0^{ls}) e^2 \leq M^s + \bar{d}e.$$

That is, for the strong agent the price constraint is always tighter than the deviation constraint, while for the weak agent it is vice versa.

Both  $\bar{k}^{ls,ss}$  and  $a_0^{sl} + a_0^{ls}$  are stricly decreasing in  $\lambda/\alpha\sigma^2$  with limits

$$\lim_{\lambda/\alpha\sigma^2 \rightarrow 0} \bar{k}^{ls,ss} = \lim_{\lambda/\alpha\sigma^2 \rightarrow 0} (a_0^{sl} + a_0^{ls}) = 1$$

and

$$\lim_{\lambda/\alpha\sigma^2 \rightarrow \infty} \bar{k}^{ls,ss} = 0 \text{ and } \lim_{\lambda/\alpha\sigma^2 \rightarrow \infty} (a_0^{sl} + a_0^{ls}) = \dots$$

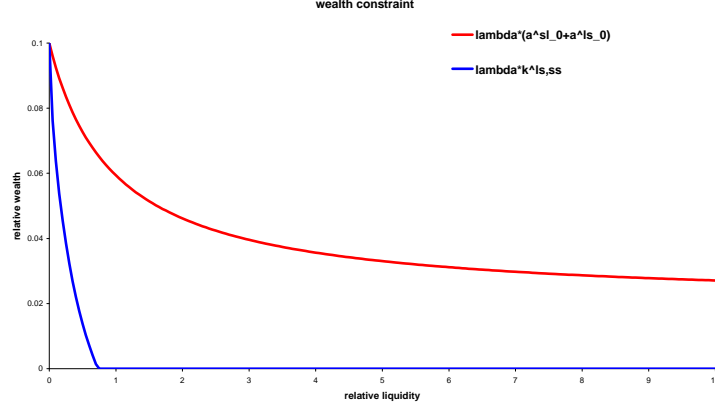


Figure 4: Two constraints for portfolio wealth of the strong and weak agents when one remains solvent and the other liquidates.

On Figure 4 we plot the two constraints of this inequality. The blue line represents the deviation constraint ( $\lambda \bar{k}^{ls,ss}$ ) and the red line stands for the price constraint ( $\lambda (a_0^{sl} + a_0^{ls})$ ). The weak agent's starting wealth has to be low enough to make sure she is not willing to increase the date-0 price in order to remain solvent.

### 5.2.3 Equilibrium of type 3: Both agents are liquidated

As we noted above, the date 1 trades when both agents liquidate are

$$x_1^s = -e_0^s \text{ and } x_1^w = -e_0^w$$

hence

$$p_1 = \bar{d} - \lambda (e_0^s + e_0^w)$$

and the optimal first-round trades have to satisfy

$$\begin{aligned} \max_{x_0^i} V_0^{ll} (M_0^i, e_0^i, M_0^{-i}, e_0^{-i}) &= M_0^i + \bar{d}e_0^i - \frac{1}{2} (e_0^i, e_0^{-i}) Q_0^{ll} (e_0^i, e_0^{-i})' \\ &= M^i + \bar{d}e + (\bar{d} - p_0) x_0^i - \frac{1}{2} (e + x_0^i, e + x_0^{-i}) Q_0^{ll} (e + x_0^i, e + x_0^{-i})'. \end{aligned}$$

**Proposition 10** *The optimal trades and the market-clearing prices when both agents are liquidated are given by*

$$x_0^s = x_0^w = x_1^s = x_1^w = -\frac{1}{2}e$$

and

$$p_0 = p_1 = \bar{d} - \lambda e.$$

Double liquidation happens in equilibrium if and only if

$$\begin{aligned} 0 &\leq M^s + \bar{d}e < \lambda \bar{k}^{ll,sl} e^2 \text{ and} \\ 0 &\leq M^w + \bar{d}e < \lambda \bar{k}^{ll,ss} e^2. \end{aligned}$$

where  $\bar{k}^{ll,sl}, \bar{k}^{ll,ss}$  are functions of  $\lambda/\alpha\sigma^2$  given in Appendix D.3.1.

The above trades and prices must be consistent with violating the constraint

$$M^i + p_0 e = M^i + (\bar{d} - \lambda e) e < 0,$$

which is equivalent to

$$0 \leq M^i + \bar{d}e < \lambda e^2 \text{ for } i = s, w,$$

otherwise we could not talk about fire-sales: as long as a risk-averse trader has non-zero holdings in a risky asset, in the absence of a force, she will not close her position.

For the existence of a Nash equilibrium we also need that  $M^s + \bar{d}e \leq \lambda \bar{k}^{ll,sl} e^2$ , and  $M^w + \bar{d}e \leq \lambda \bar{k}^{ll,ss} e^2$ , where  $\bar{k}^{ll,sl}$  and  $\bar{k}^{ll,ss}$  are functions of  $\lambda/\alpha\sigma^2$ . Being aware of the solvency constraint, agents with price impact can manipulate date 0 prices with a cost if they are able to extract higher payoffs in the next trading round. In this scenario these higher payoffs are obvious: an unconstrained risk-averse agent never sells all her holdings, in this sense the forced liquidation is clearly suboptimal. Therefore if it is not too costly to increase the price in period 0 to secure solvency, the strong agent is willing to bear this cost. The price-manipulation cost is decreasing in the agent's initial wealth, hence there exists a threshold,  $\bar{k}^{ll,sl}$ , such that the strong agent will engage in price-manipulation if and only if  $M^s + \bar{d}e \geq \lambda \bar{k}^{ll,sl} e^2$ .

As both traders have price impacts, the forced liquidation and therefore the large selling pressure of the strong agent implicitly hurts the weak trader too. Therefore it might be in her interest to manipulate the date 0 price in order to "rescue" the wealthier one. This cost is again a decreasing function of the wealth of the strong agent and therefore there exists a threshold,  $\bar{k}^{ll,ls}$ , such that the weak trader prefers to make the strong solvent if and only if  $M^s + \bar{d}e \geq \lambda \bar{k}^{ll,ls} e^2$ . If this manipulation is not too costly, the weak agent might also want to increase the date-0 price high enough to secure her solvency (and implicitly the strong agent's solvency too). This cost will be a decreasing function of her own wealth and therefore there exists a threshold,  $\bar{k}^{ll,ss}$ , such that the weak trader prefers to make both agents solvent if and only if  $M^w + \bar{d}e \geq \lambda \bar{k}^{ll,ss} e^2$ .

The above constraints yield that we have a price constraint and two deviation constraints on the wealth of the strong agent, and a price- and a deviation constraint on the wealth of the weak agent. For the existence of an equilibrium we need the portfolio wealth of the strong trade to satisfy

$$0 \leq M^s + \bar{d}e < \lambda \min \left\{ 1, \bar{k}^{ll,ls}, \bar{k}^{ll,sl} \right\} e^2,$$

while for the weak agent we must have

$$0 \leq M^w + \bar{d}e < \lambda \min \left\{ 1, \bar{k}^{ll,ss} \right\} e^2.$$

It is easy to show that  $\bar{k}^{ll,ls}, \bar{k}^{ll,sl}, \bar{k}^{ll,ss} \leq 1$  which means that the deviation constraint is always tighter than the price requirement. In fact, when agents liquidate, they smooth it through two periods to minimize their price impact and hence sell the same amount, i.e. half of their endowments in both periods. As they start with relative low but nonnegative portfolio wealths, the market-clearing price is below but is not far away from the expected payoff of the risky asset and thus the cost of driving the price up to satisfy the solvency constraint is low. Also we can show that

$$\bar{k}^{ll,sl} < \bar{k}^{ll,ls},$$

which is equivalent to saying that it is always cheaper for the strong agent to reduce her trading speed in order to stay solvent than for the weak agent to rescue her. Given these inequalities we can sum up the above constraints as

$$0 \leq M^s + \bar{d}e < \lambda \bar{k}^{ll,sl} e^2 \text{ and } 0 \leq M^w + \bar{d}e < \lambda \bar{k}^{ll,ss} e^2. \quad (10)$$

On Figure 5 we plot  $\bar{k}^{ll,sl}$  (with thin line) and  $\bar{k}^{ll,ss}$  (with the thick line) as functions of  $\lambda/\alpha\sigma^2$ . Both  $\bar{k}^{ll,sl}$  and  $\bar{k}^{ll,ss}$  are strictly decreasing in  $\lambda/\alpha\sigma^2$  with limits

$$\lim_{\lambda/\alpha\sigma^2 \rightarrow 0} \bar{k}^{ll,sl} = \lim_{\lambda/\alpha\sigma^2 \rightarrow 0} \bar{k}^{ll,ss} = 0,$$

and  $\bar{k}^{ll,ss} \geq 0$  if and only if  $\lambda/\alpha\sigma^2 \leq 0.65$ . As long as  $\lambda/\alpha\sigma^2 \leq 0.65$ , both of them are positive and hence we must have Equation 10 be satisfied by the starting wealths. For  $\lambda/\alpha\sigma^2 \geq 0.65$ ,  $\bar{k}^{ll,ss} < 0$ , hence the weak agent is always better off manipulating the price with her date-0 trade in order to make both of them solvent and thus there is no equilibrium with both agents getting liquidated.

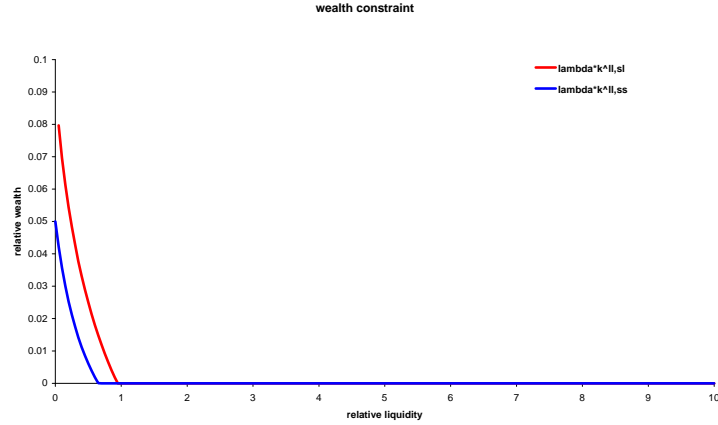


Figure 5: Two constraints on initial portfolio wealths of the strong (between the thin line and the  $x$  axis) and the weak (between the thick line and the  $x$  axis) agents as functions of  $\lambda/\alpha\sigma^2$  when both traders liquidate.

### 5.3 Existence of equilibria

Fixing the relative market depth ratio,  $\lambda/\alpha\sigma^2$ , we can examine the existence of equilibria as a function of the initial portfolio wealth of the strong and weak agents. Given that the equilibrium in which both agents are liquidated only exists if  $\lambda/\alpha\sigma^2 \leq 0.65$ , we got the following qualitatively different cases.

- If  $\lambda/\alpha\sigma^2 \leq 0.65$ , there exist all three types of equilibrium. We plot this case on

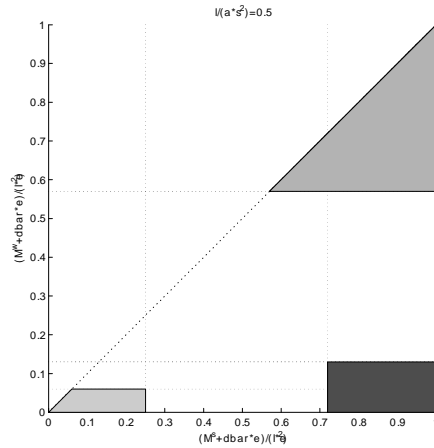


Figure 6: The three types of equilibria plotted as a function of initial wealths for  $\lambda/\alpha\sigma^2 = 0.5$ . The  $x$  axis stands for  $M^s + \bar{d}_e$ , the  $y$  axis represents  $M^w + \bar{d}_e$ . The top right area represents equilibria when both agents remain solvent, the bottom right grey area stands for equilibria with the weak agent's fire-sale, and the bottom left area shows equilibria with double liquidation.

Figure 6. On the  $x$  axis we have the initial wealth of the strong agent,  $M^s + \bar{d}_e$ , on the  $y$  axis we have the starting wealth of the weak agent,  $M^w + \bar{d}_e$ . As we



have assumed  $M^w + \bar{d}e \leq M^s + \bar{d}e$ , we only plot the three types of equilibria as a function of the initial portfolio wealths in the bottom right triangle. The area highlighted with medium grey represents equilibria when both agents remain solvent, i.e.,  $M^i + \bar{d}e \geq 2\lambda a_0^{ss} e^2$  for  $i = s, w$ . The area highlighted by dark grey stands for equilibria with the weak agent's fire-sale, i.e. when  $M^s + \bar{d}e \geq \lambda k^{ls, ll} e^2 > \lambda (a_0^{sl} + a_0^{ls}) e^2 > M^w + \bar{d}e \geq 0$ . Finally the light grey area represents equilibria with double liquidation; this happens when  $0 \leq M^i + \bar{d}e < \lambda e^2$  for  $i = s, w$ .

- If  $0.65 < \lambda/\alpha\sigma^2 \leq 0.71$ , there exist the ss and sl equilibria only. We plot this case on

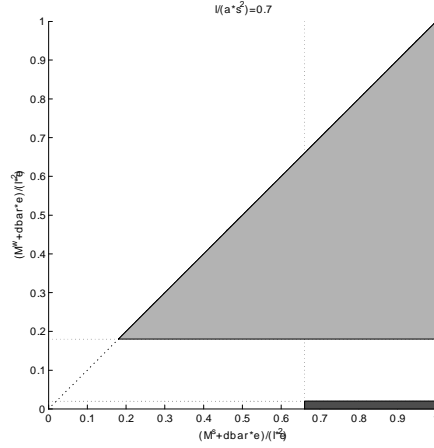


Figure 7: The two types of equilibria plotted as a function of initial wealths for  $\lambda/\alpha\sigma^2 = .7$ . The  $x$  axis stands for  $M^s + \bar{d}e$ , the  $y$  axis represents  $M^w + \bar{d}e$ . The top right area represents equilibria when both agents remain solvent, the bottom right are stands for equilibria with the weak agent's fire-sale.

Figure 7. On the  $x$  axis we have the initial wealth of the strong agent,  $M^s + \bar{d}e$ , on the  $y$  axis we have the starting wealth of the weak agent,  $M^w + \bar{d}e$ . As we assumed  $M^w + \bar{d}e \leq M^s + \bar{d}e$ , we only plot the three types of equilibria as a function of the initial portfolio wealths in the bottom right triangle. The area highlighted medium grey represents equilibria when both agents remain solvent, i.e.,  $M^i + \bar{d}e \geq 2\lambda a_0^{ss} e^2$  for  $i = s, w$ . The area highlighted by dark grey stands for equilibria with the weak agent's fire-sale, i.e. when  $M^s + \bar{d}e \geq \lambda k^{ls, ll} e^2 > \lambda (a_0^{sl} + a_0^{ls}) e^2 > M^w + \bar{d}e \geq 0$ .

- If  $0.75 < \lambda/\alpha\sigma^2$ , there exist only the ss equilibria.

We plot this case on Figure 8. On the  $x$  axis we have the initial wealth of the strong agent,  $M^s + \bar{d}e$ , on the  $y$  axis we have the starting wealth of the weak agent,  $M^w + \bar{d}e$ . As we assumed  $M^w + \bar{d}e \leq M^s + \bar{d}e$ , we only plot the three types of equilibria as a function of the initial portfolio wealths in the bottom right triangle. The area highlighted with grey represents the only equilibria, that is when both agents remain solvent, i.e.,  $M^i + \bar{d}e \geq 2\lambda a_0^{ss} e^2$  for  $i = s, w$ .

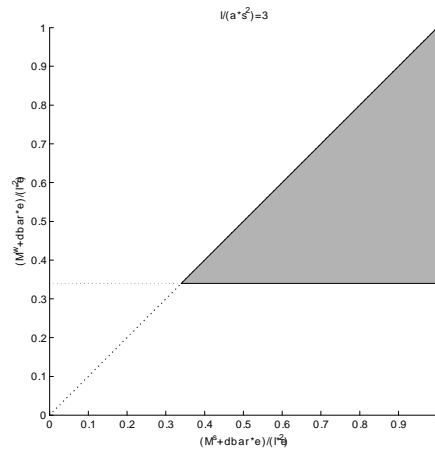


Figure 8: The two types of equilibria plotted as a function of initial wealths for  $\lambda/\alpha\sigma^2 = 3$ . The  $x$  axis stands for  $M^s + \bar{d}e$ , the  $y$  axis represents  $M^w + \bar{d}e$ . The top right area represents equilibria when both agents remain solvent.

## 6 Conclusion

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