Individual Stability in Hedonic Coalition Formation

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Abstract

I study the existence of Nash-stable and individually stable coalition structures in hedonic coalition formation games, where a coalition structure is a partition of the players, and players strictly rank the coalitions to which they belong. The hedonic coalition formation game in which all coalitions are feasible is viewed as a general model, and more specific models, such as the marriage and roommate models, are defined by the set of feasible coalitions, which are assumed to include all singletons. In this paper I give characterizations, in terms of restrictions on feasible coalitions, of the hedonic coalition formation models which are Nash-stable and individually stable, respectively, in the sense that at least one such coalition structure exists for all preference profiles in the given model. In particular, the result for Nash-stability is that coalitions of size two cannot be feasible, while for individual stability odd single-lapping cycles, together with certain disjoint subsets of the coalitions in the cycle, are not feasible. Based on these characterizations, sufficient conditions for the Nash-stability and individual stability of preference profiles are also provided.

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1 Introduction

In this paper I study hedonic coalition formation games, which were first investigated by Bogomolnaia and Jackson (2002) and Banerjee, Konishi, and Sönmez (2001). A coalition formation game is hedonic if it is given by ordinal preferences specified over coalitions for each player. Each player is a member of one coalition only, and thus a coalition structure partitions the set of players. Moreover, each player cares only about the coalition they join, so preferences for each player are over the coalitions that they are a member of, rather than over the entire coalition structure. I will assume in this paper that preference orderings are strict.

In its most general form, a hedonic coalition formation model allows for any coalition to form, and thus it is a generalization of the marriage and roommate models.[1] In the well-known marriage model the players are partitioned into two groups: the set of men and the set of women, and the set of feasible coalitions consists of all the pairs of one man and one woman, as well as all the singletons. In the roommate model, the set of feasible coalitions is larger, as it is given by all pairs and singletons. In general, a hedonic coalition formation model is defined by the set of feasible coalitions. I will assume that any set of feasible coalitions contains all the singletons, which means that it is always feasible for the players to stay alone.

In this paper I examine two stability concepts of hedonic coalition formation models which are based on individual deviations. The two concepts that I examine appear in Bogomolnaia and Jackson (2002). The first one is called *Nash-stability*.[2] Given a particular preference profile, a coalition structure is Nash-stable if no individual has an incentive to leave her current coalition and join another one that exists in the particular coalition

[1]See Roth and Sotomayor (1990) for an overview.

[2]This concept is also known as free mobility in the local public goods literature.

structure (or the empty set). The second concept is called *individual stability.* Given a particular preference profile, a coalition structure is individually stable if there is no individual and a coalition S which is either in the coalition structure or the empty set, such that the coalition which consists of this individual and the members of S, if any, is strictly preferred to their current coalition by all members of this coalition. Individual stability is weaker than Nash stability. Given that preferences are strict, this notion is also a weakening of the weak core (defined by strict domination), since it rules out blocking coalitions of the type where one individual joins an already existing coalition in the coalition structure as well as singleton blocking coalitions. Both of the above stability conditions satisfy individual rationality, which means that there is no individual who strictly prefers becoming single to staying in her coalition in the coalition structure.[3]

Why do I consider these individual stability concepts? While the core concept is a very natural concept for hedonic coalition formation games, core stable coalition structures often do not exist for preference profiles, even when very strict preference restrictions are imposed (for details see Bogomolnaia and Jackson (2002) and Banerjee, Konishi, and Sönmez (2001)). Thus, I consider these notions as meaningful alternatives to the core (in fact, individual stability is a weakening of core stability), which may lead to more positive results. These concepts may be applicable in coalition formation contexts in which forming arbitrary new coalitions may be costly or may require complex coordination among the players. If the coalitions that tend to form in some context are large, or if information on the preferences of other players is scarce (something that is not incorporated in the model considered in this paper), then considering the actions of individual players only may be quite compelling.

[3]Another stability concept that is considered in Bogomolnaia and Jackson (2002), called *contractual individual stability,* requires that each member of the coalition releasing the individual also strictly prefers that the individual leave, a property which does not satisfy individual rationality. Bogomolnaia and Jackson (2002) showed that for each preference profile there exists at least one coalition structure which satisfies this property.

In this paper I study first coalition formation models, that is, I address questions regarding sets of feasible coalitions, as opposed to preference profiles. A similar approach was taken by Pápai (2004). I call a hedonic coalition formation model Nash-stable if for each preference profile for this model there exists a Nash-stable coalition structure. Similarly, I call a hedonic coalition formation model individually stable if for each preference profile of this model there exists an individually stable coalition structure. The two main results of this paper give characterizations of coalition formation models which are Nash-stable and individually stable, respectively. In the last part of the paper, I apply these characterization results to preference profiles and find sufficient conditions for the existence of Nash-stable and individually stable coalition structures in terms of preference restrictions.

2 Hedonic Coalition Formation Models

There is a **finite set of players** $N = \{1, \ldots, n\}$. Each nonempty subset $S \subseteq N$, $S \neq \emptyset$ is a **coalition**. Let $|S|$ denote the cardinality of S. For all coalitions $S \subseteq N$, let $[S] = \{\{i\} : i \in S\}$ denote the set of singletons for the members of S. Let $\Pi = \{S \subseteq N\ : S \neq \emptyset\}$ denote the set of all coalitions in N. I assume that it is always feasible for a player to stay alone, i.e., players are not forced to join any coalition. A **collection of feasible coalitions** is given by $\Pi^* \subseteq \Pi$ such that $[N] \subseteq \Pi^*$. In the following, I will assume that a collection of feasible coalitions $\Pi^* \subseteq \Pi$ always contains $[N]$. Each collection of feasible coalitions defines a **(hedonic) coalition formation model**. For example, the roommate model is given by $\Pi^* = \{S \in \Pi : |S| \in \{1, 2\}\}$.

A **coalition structure** $\sigma = \{S_1, \ldots, S_k\}$, with $n \geq k \geq 1$, is a partition of N. For all $\Pi^* \subseteq \Pi$, let $\Sigma(\Pi^*)$ denote the set of all coalition structures $\sigma = \{S_1, \ldots, S_k\}$ such that for all $t \in \{1, \ldots, k\}$, $S_t \in \Pi^*$. For all $\sigma \in \Sigma(\Pi^*)$ and $i \in N$, let σ_i denote the coalition in σ that i belongs to.

For all $\Pi^* \subseteq \Pi$, let $\mathcal{R}(\Pi^*)$ denote the **set of preferences** over Π^* for each player $i \in N$. Since I assume that players only care about the coalition they join, player i's preferences $R_i \in \mathcal{R}(\Pi^*)$ strictly order all $S \in \Pi^*$ containing i, i.e., all the coalitions that player i is a member of. Preferences are strict, complete, and transitive. I will write SP_iS' to indicate strict preferences, and SR_iS' to indicate that either SP_iS' or $S = S'$. A **preference profile** is given by $R = (R_1, \ldots, R_n) \in \mathcal{R}^n(\Pi^*)$, where $\mathcal{R}^n(\Pi^*) = \mathcal{R}(\Pi^*) \times \ldots \times \mathcal{R}(\Pi^*)$ is the n-fold Cartesian product of $\mathcal{R}(\Pi^*)$. I will also refer to preference profiles as coalition formation problems, as opposed to coalition formation models.

A preference profile $\bar{R} \in \mathcal{R}^n(\Pi^*)$ is the **restriction of $\boldsymbol{R \in \mathcal{R}^n(\Pi^*)}$ to $\boldsymbol{\bar{\Pi} \subseteq \Pi^*}$** if $\bar{R}$ is such that for all $S \in \Pi^* \setminus \bar{\Pi}$, for all $i \in S$, $\{i\}P_iS$, for all $T \in \bar{\Pi}$, for all $i \in T$, $TR_i\{i\}$ if and only if $T\bar{R}_i\{i\}$, and for all $i \in N$ and S_1, $S_2 \in \bar{\Pi}$, $S_1\bar{R}_iS_2$ if and only if $S_1R_iS_2$. The restriction of R to $\bar{\Pi}$ will be denoted by $R|\bar{\Pi}$.

Given a coalition formation model $\Pi^* \subseteq \Pi$ and a preference profile $R \in \mathcal{R}^n(\Pi^*)$, a coalition $S \in \Pi^*$ is a **blocking coalition of $\boldsymbol{\sigma \in \Sigma(\Pi^*)}$ at $\boldsymbol{R}$** if for all $i \in S$, $SP_i\sigma_i$. A coalition structure σ is **core stable at $\boldsymbol{R \in R^n(\Pi^*)}$** if there does not exist a blocking coalition of σ at R. A coalition S is **individually rational at $\boldsymbol{R}$** if for all $i \in S$, $SR_i\{i\}$.

3 Individual Stability Concepts

A coalition structure is Nash-stable if no player prefers to become single or to join another coalition in the coalition structure.

Definition 1 *Given a coalition formation model* Π^**, a* **coalition structure $\boldsymbol{\sigma \in \Sigma(\Pi^*)}$** *is* **Nash-stable at $\boldsymbol{R \in \mathcal{R}^n(\Pi^*)}$** *if there is no* $i \in N$ *with* $\{i\}P_i\sigma_i$*, and there are no* $i \in N$ *and* $S \in \sigma$ *such that* $S \bigcup \{i\} P_i\sigma_i$*. A* **a coalition formation problem $\boldsymbol{R \in \mathcal{R}^n(\Pi^*)}$** *is* **Nash-stable** *if there exists a Nash-stable coalition structure at* R*. A* **coalition formation model $\boldsymbol{\Pi^*}$** *is* **Nash-stable** *if all* $R \in \mathcal{R}^n(\Pi^*)$ *are Nash-stable.*

Bogomolnaia and Jackson (2002) provide a sufficient condition for the Nash-stability of a preference profile. They prove that if players' preferences are additively separable and symmetric,[4] then a Nash-stable coalition structure exists.

Nash-stability is a demanding concept, since it doesn't take into account the preferences of any other player except for the one who wants to join another coalition. Individual stability is a weaker concept, as it also considers the preferences of the players in the coalition that a player wants to join. In particular, a coalition structure is individually stable if no player prefers to become single, or join another coalition in the coalition structure in which each member prefers that this player join them.

Definition 2 *Given a coalition formation model* Π^**, a* **coalition structure** $\boldsymbol{\sigma} \in \boldsymbol{\Sigma(\Pi^*)}$ *is* **individually stable at** $\boldsymbol{R} \in \boldsymbol{\mathcal{R}^n(\Pi^*)}$ *if there is no* $i \in N$ *with* $\{i\}P_i\sigma_i$ *and there are no* $i \in N$ *and* $S \in \sigma$ *such that* $S\bigcup\{i\}P_i\sigma_i$ *and, for all* $j \in S$*,* $S\bigcup\{i\}P_jS$*. A* **a coalition formation problem** $\boldsymbol{R} \in \boldsymbol{\mathcal{R}^n(\Pi^*)}$ *is* **individually stable** *if there exists an individually stable coalition structure at* R*. A* **coalition formation model** $\boldsymbol{\Pi^*}$ *is* **individually stable** *if all* $R \in \mathcal{R}^n(\Pi^*)$ *are individually stable.*

Given these two definitions, it is clear that Nash-stability implies individual stability, and both imply individual rationality, since they require that players do not prefer to become single. Also, core stability implies individual stability. This can be seen by observing that individual stability rules out certain, but not all, blocking coalitions; in particular, it rules out blocking coalitions that are formed by an existing coalition in the coalition structure together with an additional member, and it rules out singleton blocking coalitions. Note, finally, that Nash-stability and core stability are not logically related.

[4] Players' preferences are symmetric if the utility assigned by player i to player j is the same as the utility assigned by j to i.

Bogomolnaia and Jackson (2002) prove that if players' preferences are anonymous (i.e., only the size of the coalition matters, but not its members) and if preferences are single-peaked on the size of the coalition then an individually stable coalition structure exists.[5]

Further results concerning preference restrictions are given by Dimitrov and Sung (2004). They show that an individually stable coalition always exists for two subdomains of separable preferences, called aversion to enemies and appreciation of friends. In addition, they find that if preferences satisfy mutuality (friendship is always mutual) then a Nash-stable coalition structure also exists on these two domains.

4 Nash-Stable Coalition Formation Models

First I provide a characterization of Nash-stable coalition formation models. It turns out that a coalition formation model is Nash-stable if we rule out the formation of pairs, that is, coalitions with two members, and vice versa.

Theorem 1 *A coalition formation model is Nash-stable if and only if it does not have any feasible pairs.*

Proof:

If a coalition formation model is Nash-stable then it does not have any feasible pairs.

Let $\Pi^* \subset \Pi$ be Nash-stable. Suppose, by contradiction, that Π^* allows for a pair, that is, there exist two players i and j such that $\{i, j\} \in \Pi^*$. We will specify a preference profile $R \in \mathcal{R}^n(\Pi^*)$ as follows. Let i's first choice be $\{i, j\}$ followed by $\{i\}$. For every other player, including j, let the player rank being single first. Then for any possible Nash-stable coalition structure σ, we have $[N \setminus \{i\}] \subset \sigma$, which implies that $\sigma = [N]$ is the only candidate for a

[5]The Bogomolnaia and Jackson (2002) result is actually stronger; for details see Theorem 2 in their paper.

Nash-stable coalition structure at this preference profile. But $[N]$ is not Nash-stable because i prefers to join j. Thus, R is not Nash-stable, which is a contradiction.

If a coalition formation model does not have any feasible pairs then it is Nash-stable.

The proof of this direction of the theorem is trivial. All we need to note is that if no pair is feasible then the set of singletons $[N]$ is always Nash-stable, regardless of the preference profile. □

The theorem itself, though it is easy to prove and understand, is still perhaps somewhat surprising. Depending on the context, coalitions with exactly two members may be important. Most notably, the much studied marriage and roommate models only allow for the formation of pairs in addition to the singletons. Therefore, neither the marriage nor the roommate model is Nash-stable, since both of these coalition formation models have feasible pairs (in the case of the roommate model, all pairs are feasible). Similarly, many-to-one matching models are not Nash-stable either, unless being matched to a single player, for players on the 'one' side of the market, is ruled out. However, a k-sided matching model, with $k \geq 3$, is Nash-stable. k-sided matching models generalize the marriage model, as they partition the set of players into k subsets, and only allow for singletons and for coalitions with exactly k members, where each member comes from a different subset of the given partition. Thus, coalitions with two members are not feasible if k is at least three, and these models are Nash-stable by Theorem 1. Note, however, that the only Nash-stable coalition structure may be the set of singletons for any particular preference profile.

5 Individually Stable Coalition Formation Models

Nash-stability is a demanding concept, as can be seen from Theorem 1. In this section, I turn to the weaker notion of individual stability. In order to ensure that an individually stable coalition structure exists for each preference profile, we need to rule out some coalition cycles together with certain disjoint subsets of these cycles. In particular, the collection of feasible coalitions cannot contain a cycle of an odd number of coalitions, each of which has exactly one overlapping member with the two neighboring coalitions in the cycle, provided that certain subsets of the coalitions in this cycle are also feasible. We will call a set of coalitions that has this particular structure an *odd single-lapping cycle collection*.

Formally, $D \subset \Pi$ is **a single-lapping cycle collection** if

$$D = \left(\bigcup_{t=1}^{k} \bigcup_{v=2}^{|S_t|} S_t^v \right) \bigcup \left(\bigcup_{t=1}^{k} \left(\{i_t\} \bigcup S_{t+1} \right) \right),$$

where

$k \geq 3$,

$S_1, \ldots, S_k \in \Pi$ are all disjoint coalitions,

for all $t = 1, \ldots, k,\ S_t = \{i_t^1, \ldots, i_t^{|S_t|}\}$ with $|S_t| \geq 2$,

for all $t = 1, \ldots, k,\ v = 2, \ldots, |S_t|,\ S_t^v = \{i_t^1, \ldots, i_t^v\}$,

for all $t = 1, \ldots, k,\ i_t \in S_t$,

$S_{k+1} := S_1$.

If k is odd then D is an **odd single-lapping cycle collection**, and if k is even then D is an **even single-lapping cycle collection**.

In order to illuminate the structure of a single-lapping cycle collection, I introduce some more definitions here that will also be useful later. A single-lapping cycle (of coalitions) is a set of coalitions that have exactly one member in common with each of their two neighbors, and have no common member with any non-neighboring coalitions in the cycle.

$C \subset \Pi$ is a **single-lapping cycle** if $C = \{T_1, \ldots, T_k\}$ such that $k \geq 3$ and, for all $t, t' = 1, \ldots, k$ with $t < t'$, modulo k, if $t + 1 = t'$ then $|T_t \bigcap T_{t'}| = 1$, and if $t + 1 \neq t'$ then $T_t \bigcap T_{t'} = \emptyset$. (For $k = 3$, we also need to specify that $T_1 \bigcap T_2 \bigcap T_3 = \emptyset$.) If k is odd then C is an **odd single-lapping cycle** and if k is even then C is an **even single-lapping**

cycle.[6]

The single-lapping cycle in the definition of a single-lapping cycle collection above is $\{\{i_t\} \bigcup S_{t+1}\}_{t=1,\ldots,k}$. I will refer to the single common member of two neighboring coalitions in a single-lapping cycle as a *pivotal player.* Here the pivotal players are i_t, for $t = 1, \ldots, k$.

Fix $\Pi^* \subseteq \Pi$. A coalition $S \in \Pi^*$ is **strong in $\boldsymbol{\Pi^*}$** if $S = \{i_1, \ldots, i_{|S|}\}$ such that for all $v = 1, \ldots, |S| - 1,\ S_v = \{i_1, \ldots, i_v\}$ is also in Π^*. Note that all singletons are strong in any collection of coalitions Π^*, and all pairs in Π^* are also strong.

Fix $R \in \mathcal{R}^n(\Pi^*)$. A coalition $S \in \Pi^*$ is **strong at $\boldsymbol{R}$** if $S = \{i_1, \ldots, i_{|S|}\}$ such that for all $v = 1, \ldots, |S|-1$, for all $j \in S_v = \{i_1, \ldots, i_v\},\ S_v \bigcup\{i_{v+1}\} P_j S_v$, and $S_v \bigcup\{i_{v+1}\} P_{i_{v+1}} \{i_{v+1}\}$.

Intuitively, a single-lapping cycle collection in Π^* consists of $k \geq 3$ disjoint strong coalitions in Π^* together with a single-lapping cycle in Π^*, which is created by taking an arbitrary member (the pivotal player) of each of the k strong coalitions and adding this member to the next strong coalition in the cycle.

Example 1 *An odd single-lapping cycle collection.*

Let $n = 6$ and let $S_1 = \{2\}$, $S_2 = \{3, 4\}$, $S_3 = \{1, 5, 6\}$ be the disjoint strong coalitions in D. Note that $k = 3$. Let the pivotal players be $i_1 = 2$, $i_2 = 4$, $i_3 = 1$. Then the odd single-lapping cycle in D is given by $C = \{\{1, 2\}, \{2, 3, 4\}, \{1, 4, 5, 6\}\}$. Since S_t, for $t = 1, 2, 3$, are strong coalitions in D, D has to contain certain subsets of these coalitions. Given that singletons are automatically in any feasible collection of coalitions, they are not included in D, and thus we only have to include a subset of S_3 in D. Let $S_3^2 = \{1, 5\}$. Then the odd single-lapping cycle collection is given by $D = \{\{3, 4\}, \{1, 5, 6\}, \{1, 5\}, \{1, 2\}, \{2, 3, 4\}, \{1, 4, 5, 6\}\}$. $\diamondsuit$

[6]Single-lapping cycles also appeared in Pápai (2004) where they were prohibited (both odd and even), together with all pairs of coalitions with more than one member in common, in a characterization of hedonic coalition formation models that have a unique core stable coalition structure at each preference profile.

The main result of this paper, a characterization of individually stable coalition formation models, is stated below.

Theorem 2 *A coalition formation model is individually stable if and only if it does not have any feasible odd single-lapping cycle collection.*

One direction of the theorem states that an individually stable coalition formation model does not allow for an odd single-lapping cycle collection. The intuition for this (easier) direction is that if the preferences of the pivotal payers are such that they prefer coalitions in an odd single-lapping cycle in a cyclical manner, and rank the feasible subsets in the odd single-lapping cycle collection below in the order of decreasing size, then no individually stable coalition structure exists at the resulting preference profile. I will illustrate this by using the odd single-lapping cycle collection in Example 1.

Example 1 (continued) *A preference profile that is not individually stable.*

I will specify a preference profile R by listing coalitions for each player in the order of preference.

$R_1 : \{1,2\}, \{1,4,5,6\}, \{1,5,6\}, \{1,5\}, \{1\}$

$R_2 : \{2,3,4\}, \{1,2\}, \{2\}$

$R_3 : \{2,3,4\}, \{3,4\}, \{3\}$

$R_4 : \{1,4,5,6\}, \{2,3,4\}, \{3,4\}, \{4\}$

$R_5 : \{1,4,5,6\}, \{1,5,6\}, \{1,5\}, \{5\}$

$R_6 : \{1,4,5,6\}, \{1,5,6\}, \{6\}$

An inspection of this preference profile reveals that there is no coalition structure that is individually stable. $\diamondsuit$

The proofs of both directions of the theorem are given in the next section.

What are the implications of Theorem 2 for specific hedonic coalition formation models?

Since marriage and roommate models only allow for coalitions of size two, a single-lapping cycle collection is simply a single-lapping cycle for these models. But in these models all cycles are single-lapping, so ruling out odd single-lapping cycles reduces to ruling out odd cycles. The marriage model's bipartite structure does exactly that, and thus the marriage model is individually stable. On the other hand, roommate models allow for odd cycles, and thus they are not individually stable. Many-to-one matching models (assuming strict preferences over sets of players on the 'many' side of the market as well for all players) are not individually stable either because they allow for odd single-lapping cycles and for appropriate subsets of the coalitions in those cycles. Finally, k-sided matching models with $k \geq 3$ are individually stable since the lack of pairs rules out odd single-lapping cycle collections as well, as expected, given that Nash-stability implies individual stability.

6 Proof of Theorem 2

I will start with the proof of the easier direction of the theorem, which follows the intuition gained from the example in the previous section.

Necessity: *If a coalition formation model is individually stable then it does not have any feasible odd single-lapping cycle collection.*

Let $\Pi^* \subseteq \Pi$ be individually stable. Suppose, by contradiction, that there exists an odd single-lapping cycle collection $D \subset \Pi^*$. Let

$$D = \left(\bigcup_{t=1}^{k} \bigcup_{v=2}^{|S_t|} S_t^v \right) \bigcup \left(\bigcup_{t=1}^{k} \left(\{i_t\} \bigcup S_{t+1} \right) \right),$$

as specified in the definition of a single-lapping cycle collection, where $k \geq 3$ is odd. I will construct a preference profile $R \in \mathcal{R}^n(D \bigcup [N])$ and show that it is not individually stable. Since $D \bigcup [N] \subseteq \Pi^*$, this leads to a contradiction. Let R be such that, for all $t = 1, \ldots, k$, S_t are strong at R. Moreover, for all $t = 1, \ldots, k$, modulo k, let the top-ranked coalition for i_{t-1} be $\{i_{t-1}\}\bigcup S_t$. Finally, for all $t = 1, \ldots, k$, modulo k, and for all $j \in S_t$, let $\{i_{t-1}\}\bigcup S_t P_j S_t$. Let σ be individually stable at R. By feasibility, there exists $t \in \{1, \ldots, k\}$ such that $\{i_t\}\bigcup S_{t+1} \notin \sigma$. Then, given that σ is individually stable at R, it must be the case that $\{i_{t-1}\}\bigcup S_t \in \sigma$. This implies that $\{i_{t-2}\}\bigcup S_{t-1} \notin \sigma$, since it is not feasible. Therefore, $\{i_{t-3}\}\bigcup S_{t-2} \in \sigma$. Continuing the same argument, given that k is odd, $\{i_{t+2}\}\bigcup S_{t+3} \in \sigma$. Then $\{i_{t+1}\}\bigcup S_{t+2} \notin \sigma$, since it is not feasible. Hence, since S_{t+1} is strong at R and $\{i_t\}\bigcup S_{t+1}$ is not feasible, $S_{t+1} \in \sigma$. Since $\{i_t\}\bigcup S_{t+1} P_{i_t} \{i_{t-1}\}\bigcup S_t$ and, for all $j \in S_{t+1}$, $\{i_t\}\bigcup S_{t+1} P_j S_{t+1}$, this means that σ is not individually stable. This completes the proof of the necessity statement.

Sufficiency: *If a coalition formation model does not have any feasible odd single-lapping cycle collection then it is individually stable.*

For all $\Pi^* \subseteq \Pi$, for all $R \in \mathcal{R}^n(\Pi^*)$, let $F(R)$ denote the set of strong coalitions at R. Note that $[N] \subseteq F(R)$ for all R.

Lemma 1 *Fix $\Pi^* \subseteq \Pi$ and $R \in \mathcal{R}^n(\Pi^*)$. Let $\hat{R} = R|F(R)$. Then if $\sigma \in \Sigma(F(R))$ is individually stable at $\hat{R}$ then σ is also individually stable at R.*

Proof: Suppose that σ is individually stable at $\hat{R}$ but not at R. Then, since for all $j \in N$, $\sigma_j R_j \{j\}$, there exist $i \in N$ and $S \in \sigma$ such that $S\bigcup\{i\} P_i \sigma_i$ and, for all $j \in S$, $S\bigcup\{i\} P_j S$. Since σ is individually stable at $\hat{R}$, $S\bigcup\{i\} \notin F(R)$. Since $S \in \sigma$, $S \in F(R)$. Moreover, since $\sigma_i R_i \{i\}$ and $S\bigcup\{i\} P_i \sigma_i$, we have $S\bigcup\{i\} P_i \{i\}$. Since, for all $j \in S$, $S\bigcup\{i\} P_j S$ and $S \in F(R)$, this means that $S\bigcup\{i\} \in F(R)$, a contradiction. □

Let $S, T \in \Pi^*$ such that $|S \bigcap T| = 1$ and $S \neq T$. Let $S \bigcap T = \{i\}$. We will say that T **supersedes $\boldsymbol{S}$ at $\boldsymbol{R} \in \boldsymbol{\mathcal{R}^n(\Pi^*)}$** if $T P_i S$ and, for all $j \in T \setminus \{i\}$, $T P_j T \setminus \{i\}$. Let $(S, T) \in s(R)$ denote that T supersedes S at R. We will say that $E \subseteq \Pi^*$ is **s-connected at $\boldsymbol{R} \in \boldsymbol{\mathcal{R}^n(\Pi^*)}$** if, for all distinct pairs $S, T \in E$, either $(S, T) \in s(R)$, or there exist

$S_1, S_2, \ldots, S_t \in E$ with $t \geq 1$ such that $(S, S_1) \in s(R), (S_1, S_2) \in s(R), \ldots, (S_t, T) \in s(R)$.

A coalition $S \in \Pi^*$ is **top-strong at $\boldsymbol{R \in \mathcal{R}^n(\Pi^*)}$** if it is strong at R and there is no $i \in N \setminus S$ such that $S \bigcup \{i\}$ is strong at R. For all $\Pi^* \subseteq \Pi$, for all $R \in \mathcal{R}^n(\Pi^*)$, let $F^T(R)$ denote the set of top-strong coalitions at R. Note that, for all R, $F^T(R) \subseteq F(R)$.

Fix $\bar{\Pi} \subseteq \Pi$. Let $\bar{\Pi}^{\perp} := [N] \bigcup \{T' \in \Pi :$ either $T' \in \bar{\Pi}$ or there exist $S, T \in \bar{\Pi}$ such that $|S \bigcap T| = 1$ and $T' = T \setminus \{i\}$, where $S \bigcap T = \{i\}\}$.

Let $\Pi^* \subseteq \Pi$ and $R \in \mathcal{R}^n(\Pi^*)$. For all $\bar{N} \subseteq N$, let $R(\bar{N})$ denote the **restriction of $\boldsymbol{R}$ to $\boldsymbol{\bar{N}}$** such that $R(\bar{N}) = R|\bar{\Pi}$, where $\bar{\Pi} = \{S \in \Pi^* : S \subseteq \bar{N}\}$.

Lemma 2 *Fix $\Pi^* \subseteq \Pi$ and $R \in \mathcal{R}^n(\Pi^*)$. If R is not individually stable then there exists $E \subset F(R)$, $E \neq \emptyset$, such that E is s-connected at R and $R|E^{\perp}$ is not individually stable.*

Proof: Suppose that for all $E \subset F(R)$ such that E is s-connected at R, $R|E^{\perp}$ is individually stable. We will show that then R is individually stable. Let $\hat{R} = R|F(R)$. We will construct $\sigma \in \Sigma(F(R))$ that is individually stable at $\hat{R}$.

Fist we introduce an iterative procedure, which we will call the **s-top procedure**, that identifies coalitions that are potentially part of an individually stable coalition structure. Find $S_1 \in F^T(R)$ such that there is no $T \in F^T(R)$ with $(S, T) \in s(R)$, if there is any. Next, find $S_2 \in F^T(R(N \setminus S_1))$ such that there is no $T \in F^T(R(N \setminus S_1))$ with $(S_2, T) \in s(R)$. And so on, the procedure ends when there is no coalition left that is not superseded by any other coalition in the iteratively reduced set of coalitions. We will refer to the resulting coalitions, $S_1, S_2, \ldots$, as **s-top coalitions**.

Let $\bar{F} \subseteq F(R)$ be the remaining set of coalitions, which have no members in common with any of the removed s-top coalitions, and let $\bar{N}$ be the remaining set of players. Assuming that $\bar{F} \neq \emptyset$, since each coalition is superseded by some other coalition in $\bar{F}^T(R(\bar{N}))$ there exists at least one set $E \subseteq \bar{F}^T(R(\bar{N}))$, $E \neq \emptyset$, such that E is s-connected at R. Fix one of these sets. Since $E \subseteq F(R)$, $R|E^{\perp}$ is individually stable, by assumption. Therefore, since $\bar{F}^T(R(\bar{N}))$ contains only top-strong coalitions at $R(\bar{N})$, we can find $E' \subset E$ such that all coalitions are disjoint in E' and, for all $T \in E \setminus E'$ there exists $S \in E'$ such that $T \bigcap S \neq \emptyset$ and $(S, T) \notin s(R)$. Now consider the set of players remaining after all coalitions containing players in E' are removed from $\bar{F}$, and repeat the s-top procedure, followed by finding coalitions, as above, in an s-connected set. Alternating these two steps, eventually we run out of players, since N is finite. The coalition structure $\sigma \in F(R)$, which consists of all the s-top coalitions that we found iteratively and all the coalitions (as in E' above) that we identified in s-connected sets, is individually stable at $R|F(R)$, by construction. Therefore, by Lemma 1, σ is individually stable at R. □

We will call $C \subset \Pi$ an **s-cycle at *R*** if $C = \{S_1, \ldots, S_k\}$ such that for all $t = 1, \ldots, k$, modulo k, $(S_{t-1}, S_t) \in s(R)$ and, in addition, for all $t, t' = 1, \ldots, k$, modulo k, such that $t < t'$, $S_t \bigcap S_{t'} = \emptyset$, unless $t + 1 = t'$.

Lemma 3 *Fix* $\Pi^* \subseteq \Pi$ *and* $R \in \mathcal{R}^n(\Pi^*)$. *If* R *is not individually stable then there exists an s-cycle* $C \subset F(R)$ *at* R *such that* $R|C^{\perp}$ *is not individually stable.*

Proof: Fix $R \in \mathcal{R}^n(\Pi^*)$ such that R is not individually stable. By Lemma 2, there exists $E \subset F(R)$, $E \neq \emptyset$, such that E is s-connected at R and $R|E^{\perp}$ is not individually stable. Fix such an $E \subset F(R)$. We will number each coalition in E. Starting from an arbitrary coalition $S \in E$, we find a sequence of consecutively superseding coalitions until one of the coalitions is repeated. We number the coalitions in this order, so that $(S_1, S_2) \in s(R)$, $(S_2, S_3) \in s(R)$, and so on. Next, we find the highest numbered coalition which has an unnumbered coalition in E that supersedes it, and find a sequence of consecutively superseding coalitions starting from this coalition until an already numbered coalition is reached. Again, we number the coalitions in this sequence consecutively. If we keep repeating this procedure, all coalitions in E will be numbered, since E is s-connected.

We will specify a set of coalitions $G_1 \subset E$. We let $S_1 \in G_1$, and construct G_1 recursively as follows. Each $S \in G_1$ is the lowest numbered coalition in E such that it doesn't have a common member with any of the coalitions in G_1 that has a lower number than S, and we keep adding coalitions to G_1 in this manner until there is no coalition $T \in E \setminus G_1$ that is disjoint form each coalition in G_1. Note that all coalitions are disjoint in G_1.

Now we define a sequence of sets of coalitions, recursively, starting from G_1. For all $t \geq 2$, let $G_t = T^* \bigcup \{S \in G_{t-1} : (S, T^*) \notin s(R)\}$, where T^* is as follows:
i) if there is $T \notin G_{t-1}$ such that for all $S \in G_{t-1},\ S \bigcap T = \emptyset$, let T^* be the lowest numbered such coalition T,
ii) if there is no coalition T as specified in i), then find $T \notin G_{t-1}$ such that there exists $S \in G_{t-1}$ with $(S, T) \in s(R)$ and such that for all $S' \in G_{t-1}$, if $(S', T) \notin s(R)$ then $S' \bigcap T = \emptyset$, and let T^* be the lowest numbered such coalition T.
Since $R|E^{\perp}$ is not individually stable, a coalition T as specified in ii) always exists, and therefore the sequence $G_1, G_2, \ldots$ is infinite. Since N and thus Π^* are finite, there exist t, t' with $1 \leq t < t'$ such that $G_t = G_{t'}$. Let t, t' be the lowest such numbers for which $G_t = G_{t'}$.

Let $\bar{G} = \{S \in G_t :$ there exists $\bar{t}$ such that $t < \bar{t} < t'$ and $S \notin G_{\bar{t}}\}$. Then for all $S \in \bar{G}$, there exists a coalition $S' \in \bar{G}$ such that there is a sequence of consecutively superseding coalitions from S to S' that iteratively replace the previous coalition in the sequence from G_t to $G_{t'}$. For each coalition $S \in \bar{G}$, let this coalition be denoted by $S(t')$. Note that it is possible that for $S, S' \in \bar{G}$, such that $S \neq S'$, we have $S(t') = S'(t')$. In this case we can find a proper subset of $\bar{G}$ such that for all S, S' in this subset $S(t') \neq S'(t')$, since, starting from an arbitrary coalition $S \in \bar{G}$, the sequence $S(t'), S(S(t')), \ldots,$ contains a cycle, that is, eventually we get back to a coalition that is already in this sequence, given that, for all $S \in \bar{G},\ S(t') \in \bar{G}$. Let $\hat{G}$ denote this (possibly proper) subset of $\bar{G}$ such that for all $S, S' \in \hat{G}$, if $S \neq S'$ then $S(t') \neq S'(t')$. Now consider the sequence from G_t to $G_{t'}$ restricted to coalitions in $\hat{G}$. Denote this sequence by $\hat{G}_t, \ldots, \hat{G}_{t'}$.

Let $\hat{G} = \{T_1, \ldots, T_m\},\ m \geq 1$, such that $T_1(t') = T_2, T_2(t') = T_3, \ldots, T_m(t') = T_1$.

For all $l = 1, \ldots, m$, there exists a sequence of coalitions $T_l^1, \ldots, T_l^{q_l}$, $q_l \geq 1$, such that $(T_l, T_l^1) \in s(R), (T_l^1, T_l^2) \in s(R), \ldots, (T_l^{q_l}, T_{l+1}) \in s(R)$, and T_l^1 replaces T_l, T_l^2 replaces $T_l^1, \ldots$, T_{l+1} replaces $T_l^{q_l}$ in the sequence from $\hat{G}_t$ to $\hat{G}_{t'}$. Let $C = \bigcup_{l=1}^{m}\{T_l, T_l^1, \ldots, T_l^{q_l}\}$. Since we can order the members of C such that each coalition supersedes the previous one in this order, and the first coalition supersedes the last one, we can let $C = \{Z_1, \ldots, Z_q\}$, where the numbering of the coalitions is done in this order, i.e., for all $v = 1, \ldots, q$, modulo q, $(Z_v, Z_{v+1}) \in s(R)$.

Notice that there is no addition of coalition T^* as specified in i) in the definition of the sequence $G_1, G_2, \ldots$, as we move from $\hat{G}_t$ to $\hat{G}_{t'}$, because this would mean that there exist $S, S' \in \hat{G}$ such that $S \neq S'$ and $S(t') = S'(t')$. But then for all $l = 1, \ldots, m$, there cannot be three or more consecutive coalitions in the sequence $T_l^1, \ldots, T_l^{q_l}$ such that they each have a common member with some other, possibly different, coalition(s) in C, because this would imply that at least one of these coalitions would have to be added in the sequence from $\bar{G}_t$ to $\bar{G}_{t'}$. This implies that, for all $l = 1, \ldots, m$, $q_l \leq 2$. Therefore, given the construction of the sequence from $\bar{G}_t$ to $\bar{G}_{t'}$, in which (at most) one superseding coalition is added in each step, as specified in ii) in the definition of the sequence $G_1, G_2, \ldots$, the coalitions in C are such that, for all $v, v' = 1, \ldots, q$, where $v < v'$, modulo q, if $Z_v \bigcap Z_{v'} \neq \emptyset$ then $v + 1 = v'$. Therefore, $C \subset E \subseteq F(R)$ is an s-cycle at R. Furthermore, since $\bar{G}_t = \bar{G}_{t'}$, and thus the sequence is infinitely repeated, $R|C^{\perp}$ is not individually stable. □

Sufficiency proof: Let $\Pi^* \subseteq \Pi$ such that it is not individually stable. We need to show that Π^* contains an odd single-lapping cycle collection. Since Π^* is not individually stable, there exists $R \in \mathcal{R}^n(\Pi^*)$ that is not individually stable. Then, by Lemma 3, there exists a proper s-cycle $C \subset F(R)$ at R such that $R|C^{\perp}$ is not individually stable. Note that if C has an even number of coalitions then $R|C^{\perp}$ is individually stable, since a coalition structure that contains every other coalition in the proper s-cycle, together with the remaining singletons, is individually stable. Thus, C has an odd number of coalitions. Given that $C \subset F(R)$, this implies that Π^* contains an odd single-lapping cycle collection. This completes the proof of the sufficiency statement.

7 Preference Restrictions

I will now identify some sufficient conditions for a coalition formation problem (preference profile) to be Nash-stable or individually stable, based on the two characterization theorems for coalition formation models. An immediate consequence of Theorem 1 is the following simple, but demanding, condition for the existence of a Nash-stable coalition structure.

Proposition 1 *A coalition formation problem $R \in \mathcal{R}^n(\Pi)$ is Nash-stable if no player ranks any pair above staying single.*

As for individual stability, we need to define an odd single-lapping preference cycle first. A preference profile $R \in \mathcal{R}^n(\Pi)$ has an **odd single-lapping preference cycle** if there exists an odd single-lapping cycle $C \subset \Pi$ such that $C = \{T_1, \ldots, T_k\}$, as specified in the definition of a single-lapping cycle, for all $t = 1, \ldots, k$, T_k is individually rational at R, and for each pivotal player $i_t \in T_t$, for $t = 1, \ldots, k$, modulo k, $T_{t+1} P_{i_t} T_t$.

An intuitive sufficient condition for individual stability is the following.

Proposition 2 *A coalition formation problem $R \in \mathcal{R}^n(\Pi)$ is individually stable if it does not have an odd single-lapping preference cycle.*

Given Theorem 2, this sufficient condition can be weakened. I will say that a **single-lapping cycle C is based on a set of disjoint coalitions** $\{S_1, \ldots, S_k\}$ if $C = \bigcup_{t=1}^{k}$ $(\{i_t\} \bigcup S_{t+1})$, modulo k, where, for all $t = 1, \ldots, k$, $i_t \in S_t$. Also, I will say that a **single-lapping preference cycle is based on a set of disjoint coalitions** if there exists a single-lapping cycle $C = \bigcup_{t=1}^{k}(\{i_t\} \bigcup S_{t+1})$ that is based on a set of disjoint coalitions $\{S_1, \ldots, S_k\}$ such that, for all $t = 1, \ldots, k$, modulo k, $\{i_t\} \bigcup S_{t+1} P_{i_t} S_t$ and, for all $j \in S_{t+1}$, $\{i_t\} \bigcup S_{t+1} P_j S_{t+1}$.

Proposition 3 *A coalition formation problem $R \in \mathcal{R}^n(\Pi)$ is individually stable if it does not have a single-lapping preference cycle based on an odd number of disjoint coalitions that are strong at R.*

One may further weaken these sufficient conditions, but a characterization is more difficult. To see this, note that if all players rank N first then N is individually stable. Indeed, any unanimously top-ranked coalition structure is individually stable, and, given such a unanimous choice, it is irrelevant whether the preference profile has an odd single-lapping preference cycle. What is difficult to pin down is when to rule out such a preference cycle, since it is hard to specify when this cycle is ranked "high" enough so that it becomes relevant and makes it impossible to have an individually stable coalition structure. Such a specification would most likely be complicated, and thus there is a trade-off between the simplicity and the weakness of a sufficiency condition.

The sufficiency conditions given here are different from the ones provided by Bogomolnaia and Jackson (2002), as these conditions are based on the specific solution concepts, instead of using standard preference restrictions which invariably turn out to be too strong. When the sufficient conditions for individual stability given in this paper are compared to a standard condition, such as anonymity combined with single-peaked preferences over the size of coalitions, it is apparent that the "no-odd-cycles" type of conditions are generally less prohibitive and thus more appropriate in this context.

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