Stable Allocations of Risk

Péter Csóka* P. Jean-Jacques Herings† László Á. Kóczy‡

June 29, 2007

Abstract

Measuring risk can be axiomatized by the concept of coherent measures of risk. A risk environment specifies some individual portfolios' realization vectors and a coherent measure of risk. We consider sharing the risk of the aggregate portfolio by studying transferable utility cooperative games: risk allocation games.

We show that the class of risk allocation games coincides with the class of totally balanced games. As a limit case the aggregate portfolio can have the same payoff in all states of nature. We prove that the class of risk allocation games with no aggregate uncertainty coincides with the class of exact games.

Keywords: Coherent Measures of Risk, Risk Allocation Games, Totally Balanced Games, Exact Games

JEL Classification: C71

1 Introduction

Financial institutions are regulated to hold economic capital as a cushion against default. A regulator determines the institution's required economic capital by a measure of risk. Risk measures are also used to calculate portfolio managers' risk adjusted performance. It is therefore crucial to measure and as there is usually a diversification effect allocate risk in an appropriate way.

---

*Department of Economics, Universiteit Maastricht, P.O. Box 616, 6200 MD, Maastricht, The Netherlands. E-mail:P.Csoka@algec.unimaas.nl.

†Department of Economics, Universiteit Maastricht, P.O. Box 616, 6200 MD, Maastricht, The Netherlands. E-mail:P.Herings@algec.unimaas.nl. The author would like to thank the Netherlands Organisation for Scientific Research (NWO) for financial support.

‡Department of Economics, Universiteit Maastricht, P.O. Box 616, 6200 MD, Maastricht, The Netherlands. E-mail:L.Koczy@algec.unimaas.nl. The author would like to thank funding by the Netherlands Organisation for Scientific Research (NWO) and by the European Union under the Marie Curie Intra-European Fellowship MEIF-CT-2004-011537.

We will use the term *portfolio* when referring to a risky entity (portfolio of stocks and bonds, firm, insurance company, bank, etc.). The value of a portfolio might change due to all kinds of uncertain events. For the sake of simplicity we use realization vectors of discrete random variables, specifying the portfolios payoff in all states of nature. A *measure of risk* assigns a real number to a realization vector. It is the minimal amount of cash the regulated agent has to add to his portfolio, and to invest in a zero coupon bond for its risk to be acceptable to the regulator. *Coherent measures of risk* (Artzner, Delbaen, Eber, and Heath, 1999) are defined by four axioms: monotonicity, subadditivity, positive homogeneity and translation invariance. In particular, they include *spectral measures of risk* (Acerbi, 2002), for instance the *discounted maximum loss*. Csóka, Herings, and Kóczy (2007b) show that the axioms of coherent measures of risk are compatible with a natural general equilibrium approach for measuring risk.

When the risk measure is subadditive then the risk of an aggregate portfolio consisting of many individual portfolios is lower than the sum of the risks of the individual portfolios. To allocate the whole risk to the subportfolios one has to share the risk diversification effects. Risk allocation is modeled by Denault (2001) with transferable utility cooperative games. We separate the *risk environment* specifying the individual portfolios' realization vectors and a coherent measure of risk from the derived cooperative game that we call *risk allocation game*.

A *totally balanced game* is a cooperative game having a non-empty core in all of its subgames. Totally balanced games arise from a wide range of applications. They coincide with market games (Shapley and Shubik, 1969); also with a special case of market games with a continuum of indivisible commodities: cooperation in fair division (Legut, 1990); they are equivalent to a class of maximum flow problems (Kalai and Zemel, 1982a); and also to permutation games of less than four players (Tijs, Parthasarathy, Potters, and Prassad, 1984). Moreover, totally balanced games are generated by linear production games (Owen, 1975), generalized network problems (Kalai and Zemel, 1982b), and controlled mathematical programming problems (Dubey and Shapley, 1982).

We show that the class of risk allocation games coincides with the class of totally balanced games, that is all risk allocation games are totally balanced and all totally balanced games can be generated by a risk allocation game with a properly specified risk environment. Using linear programming we characterize all the risk environments with the maximum loss that generate a given totally balanced game.

In many cases the individual portfolios are riskier than the aggregate portfolio. In this paper we study the limit case when there is no aggregate uncertainty, that is the payoff of the aggregate portfolio is the same in all states of nature. We show that the class of risk allocation games with no aggregate uncertainty coincides with the class of *exact games* (Schmeidler, 1972). As evidenced by the previous paragraphs, there are many applications giving rise to the class of totally balanced games. to the best of our knowledge, risk allocation with no aggregate uncertainty is the first application that leads to the class of exact games.

The structure of the paper is as follows. First we introduce coherent measures of risk, transferable utility games and risk allocation games. In Section 3 we prove that the class of risk allocation games coincides with the class of totally balanced games and investigate our constructive proof by linear programming. In Section 4 we show that the class of risk allocation games with no aggregate uncertainty coincides with the class of exact games. In Section 5 we conclude.

2 Preliminaries

2.1 Coherent Measures of Risk

Consider the set $\mathbb{R}^S$ of realization vectors, where S denotes the number of states of nature. State of nature s occurs with probability $p_s > 0$ and $\sum_{s=1}^{S} p_s = 1$. The vector $X \in \mathbb{R}^S$ represents a portfolio's (firm's, insurance company's, bank's, etc.) possible profit and loss realizations on a common chosen future time horizon, say at $t = 1$. The amount X_s is the portfolio's payoff in state of nature s. Negative values of X_s correspond to losses. The inequality $Y \geq X$ means that $Y_s \geq X_s$ for all $s = 1, \ldots, S$.

A *measure of risk* is a function $\rho : \mathbb{R}^S \to \mathbb{R}$ measuring the risk of a portfolio from the perspective of the present ($t = 0$). It is the minimal amount of cash the regulated agent has to add to his portfolio, and to invest in a reference instrument today, such that it ensures that the risk involved in the portfolio is acceptable to the regulator. We assume that the *reference instrument* has payoff 1 in each state of nature at $t = 1$, thus its realization vector is $1^S = (1, \ldots, 1)^\top$. The reference instrument is riskless in the "classical sense", having no uncertainty in its payoffs. It is most natural to think of it as a zero coupon bond. The price of the reference instrument, the *discount factor* is denoted by $\delta \in \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$. We adjust the definition of coherent measures of risk to the discrete case with realization vectors as follows.

Definition 2.1. A function $\rho : \mathbb{R}^S \to \mathbb{R}$ is called a *coherent measure of risk* (Artzner et al., 1999) if it satisfies the following axioms.

1. *Monotonicity*: for all $X, Y \in \mathbb{R}^S$ such that $Y \geq X$, we have $\rho(Y) \leq \rho(X)$.

2. *Subadditivity*: for all $X, Y \in \mathbb{R}^S$, we have $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

3. *Positive homogeneity*: for all $X \in \mathbb{R}^S, h \in \mathbb{R}_+$, we have $\rho(hX) = h\rho(X)$.

4. *Translation invariance*: for all $X \in \mathbb{R}^S$ and $a \in \mathbb{R}$, we have $\rho(X + a1^S) = \rho(X) - \delta a$.

Acerbi (2002) treats a subclass of coherent measures of risk: *spectral measures of risk* in case of discrete random variables with *equiprobable* outcomes, that is when $p_1 = \cdots = p_S = 1/S$, as a special case. He assumes that the discount factor is 1. In order to be compatible with the definition of coherent measures of risk, we have to multiply his definition by δ. The definition of spectral measures of risk with equiprobable outcomes is as follows.

Let us introduce the ordered statistics $X_{s:S}$ given by the ordered values of the S-tuple $X_1, \ldots, X_S$, that is $\{X_{1:S}, \ldots, X_{S:S}\} = \{X_1, \ldots, X_S\}$ and $X_{1:S} \leq X_{2:S} \leq \cdots \leq X_{S:S}$.

Definition 2.2. Let the outcomes be equiprobable. Consider a vector $\phi \in \mathbb{R}^S$. The measure $M_\phi : \mathbb{R}^S \to \mathbb{R}$ defined by

$$M_\phi(X) = -\delta \sum_{s=1}^{S} \phi_s X_{s:S} \tag{1}$$

is a *spectral measure of risk* if $\phi \in \mathbb{R}^S$ satisfies the conditions

1. Nonnegativity: $\phi_s \geq 0$ for all $s = 1, \dots, S$,
2. Normalization: $\sum_{s=1}^{S} \phi_s = 1$,
3. Monotonicity : ϕ_s is non-increasing, that is $\phi_{s_1} \geq \phi_{s_2}$ if $s_1 < s_2$ and $s_1, s_2 \in \{1, \dots, S\}$.

Spectral measures of risk are discounted weighted average losses, with non-increasing weights, with the highest weight on the worst outcome. The weight vector ϕ is the so-called *risk spectrum*, the "attitude" toward risk. In the following we define an example for a spectral measure of risk.

Definition 2.3. Let the outcomes be equiprobable and let $k \in \{1, \dots, S\}$. The *k-expected shortfall* of the realization vector X is defined by

$$\mathrm{ES}_k(X) = -\delta \sum_{s=1}^{k} \frac{1}{k} X_{s:S}. \tag{2}$$

The k-expected shortfall is the discounted average of the worst k outcomes. For a detailed discussion see Acerbi and Tasche (2002).

2.2 Transferable Utility Games

Let $N = \{1, \dots, n\}$ denote the finite *set of players. A value function* $v : 2^N \to \mathbb{R}$ defined on all subsets of N satisfying $v(\{\emptyset\}) = 0$ gives rise to a *cooperative game with transferable utility* (*game*, for short) (N, v). Let Γ denote the set of games with n players. A *single-valued solution concept* is a function $\Phi : \Gamma \to \mathbb{R}^n$ that maps each game to a unique *allocation* $x \in \mathbb{R}^n$, where x_i is the payoff of player $i \in N$. An allocation $x \in \mathbb{R}^n$ is called *efficient*, if $\sum_{i \in N} x_i = v(N)$; *individually rational*, if $x_i \geq v(i)$ for all $i \in N$; and *coalitionally rational* if $\sum_{i \in C} x_i \geq v(C)$ for all $C \in 2^N$. Efficient and individually rational allocations are *imputations*, their set is denoted by $\mathcal{I}$. The set of efficient and coalitionally rational allocations, the *core* is denoted by $\mathcal{C}$.

Let for each $C \in 2^N$, $a(C) \in \mathbb{R}^n$ be the membership vector in C, where $a_i(C) = 1$ if $i \in C$ and $a_i(C) = 0$ otherwise. Let $(\lambda^C)_{C \in 2^N} \in \mathbb{R}_+$ denote the *collection of numbers* $\lambda^C \in \mathbb{R}_+$ for all $C \in 2^N$.

Definition 2.4. A *balanced collection of weights* is a collection of numbers $(\lambda^C)_{C\in 2^N} \in \mathbb{R}_+$ such that $\sum_{C\in 2^N} \lambda^C a(C) = a(N)$. A game is *balanced* if $\sum_{C\in 2^N} \lambda^C v(C) \leq v(N)$ for all balanced collections of weights.

One of the well-known interpretations is that the players can distribute one unit of working time to any coalition and if each coalition is active during a fraction λ^C of a unit of time then the players cannot reach more than $v(N)$, the value of the grand coalition. Balancedness is a necessary and sufficient condition for a non-empty core in a transferable utility game (Bondareva, 1963; Shapley, 1967). See Predtetchinski and Herings (2004) for the appropriate extension of the concept of balancedness to be necessary and sufficient for non-emptiness of the core in non-transferable utility games.

For a game (N, v) and a coalition $C \in 2^N$ a *subgame* v^C is obtained by restricting v to subsets of C.

Definition 2.5. A game (N, v) is *totally balanced* if for every $D \in 2^N$ its subgame v^D is balanced, that is, if for all $D \in 2^N$ and for all collections of real numbers $(\lambda^C)_{C\in 2^D} \in \mathbb{R}_+$ satisfying $\sum_{C\in 2^D} \lambda^C a(C) = a(D)$ we have $\sum_{C\in 2^D} \lambda^C v(C) \leq v(D)$.

In a totally balanced game every subgame has a non-empty core. Let Γ_{tb} denote the family of totally balanced games with n players. An interesting subclass of totally balanced games are *exact games* (Schmeidler, 1972).

Definition 2.6. A game (N, v) is an *exact game* if for every $C \in 2^N$ there exists a core allocation $x \in \mathcal{C}$ such that $x(C) = v(C)$.

Let Γ_{e} denote the family of exact games with n players. Csóka, Herings, and Kóczy (2007a) give the following necessary and sufficient condition for a game to be exact.

Theorem 2.7. *A game $(N, v) \in \Gamma$ is exact if and only if it is totally balanced and overbalanced.*

Definition 2.8. An *overbalanced collection of weights* is a collection of real numbers $(\lambda^C)_{C\subset N} \in \mathbb{R}_+$ such that $\sum_{C\in 2^N\setminus\{D,N\}} \lambda^C a(C) = a(N) + \lambda^D a(D)$. A game is *overbalanced* if $\sum_{C\in 2^N\setminus\{D,N\}} \lambda^C v(C) \leq v(N) + \lambda^D v(D)$ for all overbalanced collections of weights.

The difference between balancedness and overbalancedness is that in case of overbalancedness one coalition is forced to exist for a non-positive amount of time $(-\lambda^D \leq 0)$, which allows players to spend more than one unit of time in the other coalitions.

Convex games are a subset of exact games (Schmeidler, 1972).

Definition 2.9. A game (N, v) is *convex* if for all $C, D \in 2^N$ we have that
$v(C) + v(D) \leq v(C \cup D) + v(C \cap D)$.

Let Γ_{c} denote family of convex games with n players. We have that $\Gamma_{\mathrm{tb}} \supseteq \Gamma_{\mathrm{e}} \supseteq \Gamma_{\mathrm{c}}$.

2.3 Risk Allocation Games

Denault (2001) introduces *risk capital allocation problems*: Suppose a firm has n portfolios, the matrix of their realization vectors[1] is given by $X \in \mathbb{R}^{S\times n}$. The question is how to allocate the risk of the firm measured by a coherent measure of risk to the portfolios.

Let $X_{\cdot i}$ denote the i-th column of X, it is the realization vector of portfolio i. Let $X_{s\cdot}$ denote the row of X corresponding to state of nature s; $X_{s,i}$ its element at row s and column i; and $\{X_{s,i}\}_{i\in D}$ the row vector corresponding to state of nature s with elements $i \in D$. For a coalition of portfolios $C \in 2^N$ let $X(C) = \sum_{i\in C} X_{\cdot i}$ and $X_s(C) = \sum_{i\in C} X_{s,i}$.

Denault (2001) assumes that the nth portfolio equals $b \in \mathbb{R}$ units of reference instrument: $X_{\cdot n} = b1^S$, but we will consider a more general setting, where X_n can be any portfolio. Moreover, we separate the risk environment from the induced game.

Definition 2.10. A *risk environment* is a tuple (n, S, p, X, ρ), where n is the number of portfolios, S indicates the number of states of nature, $p = (p_1, \ldots, p_S)$ is the vector of probabilities, X is the matrix of realization vectors, and ρ is a coherent measure of risk.

Definition 2.11. Given a risk environment (n, S, p, X, ρ) a *risk allocation game* is a game (N, v), where the value function $v : 2^N \to \mathbb{R}$ is defined by

$$v(C) = -\rho(X(C)) \text{ for all } C \in 2^N, \tag{3}$$

and $\rho(\{\emptyset\}) = 0$ by definition.

A risk allocation game with n players is induced by the number of states of nature, their probability of occurrence, n realization vectors and a coherent measure of risk. Let Γ_{r} denote the family of risk allocation games with n players. In such a game, according to Equation (3), the larger the risk of any subset of portfolios, the lower its value.

If the rows of a matrix of realization vectors sum up to the same number then it captures that there is *no aggregate uncertainty*. Formally:

Definition 2.12. *A payoff matrix $X \in \mathbb{R}^{S\times n}$ expresses no aggregate uncertainty* if there exists a number $h \in \mathbb{R}$ such that for all $s \in \{1, \ldots, S\}$ we have that $X_s(N) = h$.

Let Γ_{rn} denote the family of risk allocation games with n players with no aggregate uncertainty. Obviously, $\Gamma_{\mathrm{rn}} \subseteq \Gamma_{\mathrm{r}}$. We first study risk allocation games in general, then with no aggregate uncertainty.

3 Total Balancedness

3.1 Risk Allocation Games and Totally Balanced Games

Denault (2001) proves (Theorem 4) that the family of risk capital allocation problems is balanced (Definition 2.4). As a subgame of a risk allocation game is also a risk allocation game, we can adjust his proof to show that risk allocation games are totally balanced.

[1] Denault (2001) uses continuously distributed random variables. We adjust his setting to the more tractable setup with discrete random variables, resulting in realization vectors.

Proposition 3.1. *All risk allocation games $v \in \Gamma_r$ are totally balanced, $\Gamma_r \subseteq \Gamma_{tb}$.*

Proof. Take any risk environment (n, S, p, X, ρ). We show that for any subgame the induced risk allocation game (N, v) is balanced. Take any coalition $D \in 2^N$ and consider the subgame v^D. Take any collection of real numbers $(\lambda^C)_{C \in 2^D} \in \mathbb{R}_+$ such that $\sum_{C \in 2^D} \lambda^C a(C) = a(D)$. Then by Equation (3), the positive homogeneity and subadditivity of ρ we have that

$$\sum_{C \in 2^D} \lambda^C v^D(C) = -\sum_{C \in 2^D} \rho(\sum_{i \in C} \lambda^C X_{\cdot i}) \leq -\rho(\sum_{C \in 2^D} (\sum_{i \in C} \lambda^C X_{\cdot i})) =$$
$$= -\rho(\sum_{i \in D} (\sum_{C \in 2^D, C \ni i} \lambda^C X_{\cdot i})) = -\rho(\sum_{i \in D} X_{\cdot i}) = -\rho(X(D)) = v(D),$$

where the last line follows from rearranging the summation and using the fact that we have a balanced collection of weights. Thus the arbitrarily chosen subgame is balanced. □

Not only is it true that all risk allocation games are totally balanced, but also any totally balanced game can be generated by a risk allocation game as the following proposition shows. We illustrate Proposition 3.2and its proof by Example 3.3.

Proposition 3.2. *For each totally balanced game $(N, v) \in \Gamma_{tb}$ there exists a risk environment inducing it, so $\Gamma_{tb} \subseteq \Gamma_r$.*

Proof. Take any totally balanced game $(N, v) \in \Gamma_{\text{tb}}$. Let us zero-normalize v using

$$v_0(C) = v(C) - \sum_{i \in C} v(\{i\}). \tag{4}$$

It is easy to see that v_0 is also totally balanced. Using the singletons with weights one it follows from the total balancedness of v_0 that for any $C \in 2^N$

$$0 = \sum_{i \in C} v_0(\{i\}) \leq v_0(C). \tag{5}$$

Moreover, for any $C \in 2^N$ partitioning N into C and $N \setminus C$ we have that

$$v_0(C) + v_0(N \setminus C) \leq v_0(N). \tag{6}$$

Using Equations (5) and (6) we obtain that for any $C \in 2^N$

$$0 \leq v_0(C) \leq v_0(N). \tag{7}$$

The proof is constructive. First we consider the zero-normalized game, then we show how the construction can be generalized to the original game. Specify the risk environment (n, S, p, X^0, ρ) as follows. Let us introduce a state of nature for all non-empty coalitions of N, thus let $S = 2^n - 1$. We label states of nature by $C, D \in 2^N \setminus \{\emptyset\}$. Let $p_1 = \cdots = p_S = 1/S$, and let ρ be the 1-expected shortfall (Definition 2.3), that is for any $X \in \mathbb{R}^S$

$$\rho(X) = \mathrm{ES}_1(X) = -\delta X_{1:S}. \tag{8}$$

For the moment assume that $\delta = 1$. Then for a realization vector $X^0_{\cdot i} \in \mathbb{R}^{2^n-1}$ Equation (8) can be rewritten as

$$\rho(X^0_{\cdot i}) = -\min_{D \in 2^N} X^0_{D,i}, \tag{9}$$

so ρ simplifies to the maximum loss. The risk environment (n, S, p, X^0, ρ) induces the game $(N, \bar{v}_0)$. We will show that $\bar{v}_0 = v_0$. Using Definition 2.11 we have that for every $C \in 2^N \setminus \{\emptyset\}$,

$$\bar{v}_0(C) = -\rho(X^0(C)) = \min_{D \in 2^N} X^0_D(C). \tag{10}$$

For each coalition (state of nature) $C \in 2^N \setminus \{\emptyset\}$ let the row vector $X^0_{C\cdot}$ be such that

$$\{X^0_{C,i}\}_{i \in C} \text{ is a point in the core of the subgame } v_0^C \text{, and} \tag{11}$$

$$\text{for all } i \notin C \text{ let } X^0_{C,i} = v_0(N). \tag{12}$$

It follows from the definition of a subgame, from (11), and the efficiency of a core element that for every $C \in 2^N \setminus \{\emptyset\}$

$$v_0^C(C) = v_0(C) = X^0_C(C). \tag{13}$$

Notice, that for all $C, D \in 2^N$ we have that

$$X^0_C(C) \leq X^0_D(C), \tag{14}$$

since if $D \supseteq C$ then Inequality (14) follows from (11) as we have that for a core element $\{X^0_{D,i}\}_{i \in D}$ in subgame v^0_D

$$X^0_C(C) = v_0(C) \leq X^0_D(C), \tag{15}$$

and if $D \not\supseteq C$ then one of the components of $\{X^0_{D,i}\}_{i \in C}$ is $v_0(N)$, and using Equation (7) Inequality (14) follows immediately. Combining Equations (13) and (14) with Equation (10) we obtain that $\bar{v}_0 = v_0$.

It is easy to see that after normalizing back using the realization vectors $X_{\cdot i} = X^0_{\cdot i} + 1^S v(\{i\}), i \in N$, we can conclude that $\bar{v} = v$. After multiplying the components of X by $1/\delta$, the $\delta = 1$ assumption can also be released. □

C	$v(C)$	$v_0(C)$
$\{1\}$	-10	0
$\{2\}$	3	0
$\{3\}$	-2	0
$\{1,2\}$	-4	3
$\{1,3\}$	-6	6
$\{2,3\}$	2	1
$\{1,2,3\}$	-1	8

Table 1: A totally balanced game and its zero-normalized game.

S	$X^0_{\cdot 1}$	$X^0_{\cdot 2}$	$X^0_{\cdot 3}$	$X_{\cdot 1}$	$X_{\cdot 2}$	$X_{\cdot 3}$
$\{1\}$	0	8	8	-10	11	6
$\{2\}$	8	0	8	-2	3	6
$\{3\}$	8	8	0	-2	11	-2
$\{1,2\}$	1	2	8	-9	5	6
$\{1,3\}$	2	8	4	-8	11	2
$\{2,3\}$	8	1	0	-2	4	-2
$\{1,2,3\}$	2	1	5	-8	4	3

Table 2: Payoff matrices for the zero normalized game and for the original game.

Example 3.3. Consider the following example with 3 players, where we show how a totally balanced game can be generated by a risk allocation game.

It is easy to see that v is the value function of a totally balanced game and v_0 is obtained by zero-normalizing v. Note that Inequality (7) is satisfied by v_0.

In Table 2 we have specified the matrix of realization vectors X^0 according to requirements (11) and (12). For instance, for $C = \{1,2\}$ we have that $(X^0_{\{1,2\},1}, X^0_{\{1,2\},2}) = (1,2)$ is a point in the core of the subgame with players 1 and 2, and $X^0_{\{1,2\},3} = 8 = v_0(N)$. Assume that $\delta = 1$. It is easy to check that X^0 and the 1-expected shortfall generate v_0. The matrix X is obtained by using $X_{\cdot i} = X^0_{\cdot i} + 1^S v(\{i\})$ for all $i \in N$. As one can verify the original game v is induced by X and the 1-expected shortfall.

Note that in our constructive proof the statement of Proposition 3.2 is strengthened in the sense that the family of games induced by risk environments with $S \leq 2^n - 1$ and the 1-expected shortfall equals the family of totally balanced games with n players, that is any totally balanced game can be generated by a properly specified risk environment with the 1-expected shortfall and $2^n - 1$ (or lower) states of nature. From Propositions 3.1 and 3.2 we have the following theorem.

Theorem 3.4. *The class of risk allocation games coincides with the class of totally balanced games,* $\Gamma_r = \Gamma_{tb}$.

Interestingly, Kalai and Zemel (1982b) uses a similar construction to show that a game is totally balanced if and only if it is the minimum game of a finite collection of *additive* games. A game (N, v) is called *additive* if there exists a set of real numbers $b_1, \dots, b_n$ such that for every $C \in 2^N$, $v(C) = \sum_{i \in C} b_i$. For a collection of games $\{v_t\}_{t \in T}$ the *minimum game* is defined by $(\min v_t)(C) = \min_{t \in T} v_t(C)$. It is easy to see that the totally balanced game v in Table 1 is also generated as the minimum game of the additive games generated by $X_{C\cdot}, C \in 2^N$, in Table 2.

3.2 Linear Programming Results

Take any totally balanced game $(N, v) \in \Gamma_{\text{tb}}$. Just like in Proposition 3.2 let $S = 2^n - 1$, $p_1 = \dots = p_S = 1/S$ and let ρ be the 1-expected shortfall with $\delta = 1$ throughout the subsection. Whenever we write v is generated by a matrix of realization vectors X we mean that the risk allocation game induced by the risk environment (n, S, p, X, ρ) equals v, where the other elements of the risk environment are the ones assumed above.

In Proposition 3.2 the matrix of realization vectors X generating v was constructed using the *core requirement*[2]: for every $C \in 2^N \setminus \{\emptyset\}$

$$\{X_{C,i}\}_{i \in C} \text{ is a point in the core of the subgame } v^C. \tag{16}$$

The other elements of X were chosen to be sufficiently large.

In this subsection we investigate Proposition 3.2 by linear programming and characterize all the matrices generating v. We develop the linear programming problem such that the matrices derived from its optimal solutions (which are vectors) generate v. To do this we define the vector $\hat{X} \in \mathbb{R}^{Sn}$ by juxtaposing the rows of $X \in \mathbb{R}^{S \times n}$, that is $\hat{X} = (X_{1\cdot}, X_{2\cdot}, \dots, X_{S\cdot})^\top \in \mathbb{R}^{Sn}$. We will use the notations $\hat{X}$ and X interchangeably, depending on whether we need a vector or a matrix containing the same numbers. Using the row vector of zeros $0^n = (0, 0, \dots, 0) \in \mathbb{R}^{1 \times n}$ for every $C \in 2^N \setminus \{\emptyset\}$ we define the matrices

$$A(C) = \begin{pmatrix} a(C)^\top & 0^n & & 0^n \\ 0^n & a(C)^\top & & 0^n \\ & & \ddots & \vdots \\ 0^n & 0^n & \cdots & a(C)^\top \end{pmatrix} \in \mathbb{R}^{S \times Sn} \tag{17}$$

containing the membership vector $a(C)$ transposed along the "diagonal" and 0^n otherwise.

For a matrix $X \in \mathbb{R}^{S \times n}$, similarly to Equation (10), for every $C \in 2^N \setminus \{\emptyset\}$ the value function of the induced risk allocation game $(N, \bar{v})$ is given by

$$\bar{v}(C) = \min_{D \in 2^N} X_D(C). \tag{18}$$

[2]There we had a zero normalized game, but it is easy to see that after normalizing back the core requirement is still satisfied.

Let $A_D(C)$ denote the D-th row of $A(C)$. For a vector $\hat{X} \in \mathbb{R}^{Sn}$ Equation (18) can be rewritten as for every $C \in 2^N \setminus \{\emptyset\}$

$$\bar{v}(C) = \min_{D \in 2^N} A_D(C)\hat{X}. \tag{19}$$

Equation (19) implies that for every $C \in 2^N \setminus \{\emptyset\}$

$$A(C)\hat{X} \geq \bar{v}(C)1^S. \tag{20}$$

We introduce some more notations. Take any game $(N, v) \in \Gamma$.

Let $E = (a(\{1\})^\top, a(\{2\})^\top, \ldots, a(N)^\top) \in \mathbb{R}^{1 \times Sn}$,

$$V = \begin{pmatrix} v(\{1\})1^S \\ v(\{2\})1^S \\ \vdots \\ v(N)1^S \end{pmatrix} \in \mathbb{R}^{S^2} \text{ and} \tag{21}$$

$$A = \begin{pmatrix} A(\{1\}) \\ A(\{2\}) \\ \vdots \\ A(N) \end{pmatrix} \in \mathbb{R}^{S^2 \times Sn}. \tag{22}$$

Consider the following linear programming problem.

$$(\mathrm{P}_v) \qquad \begin{array}{c} \min E\hat{X} \\ \text{s. t.} \\ \\ A\hat{X} \geq V \\ \hat{X} \in \mathbb{R}^{Sn}. \end{array}$$

The objective function of (P_v) captures the constructive proof of Proposition 3.2, as it is minimizing exactly the sum of those elements of $\hat{X}$ which are used in the core requirement (16). To put it differently: we assign a different row of X to each coalition of N. Using Equation (20) it can be seen that the feasibility constraints are the necessary requirements for v to be generated by the matrix derived from a feasible solution.

The set of optimal solutions of (P_v) is non-empty, since $\hat{X} = (k, \ldots, k) \in \mathbb{R}^{Sn}$ is feasible solution, where $k = \max\{\max_{C \in 2^N} v(C), 0\}$ and the set of feasible solutions is bounded from below. Let $\mathcal{X}_v^*$ denote the set of optimal solutions of (P_v) and $\hat{X}^* \in \mathbb{R}^{Sn}$ its elements.

Proposition 3.5. *Take any game $(N, v) \in \Gamma$ and any optimal solution of (P_v) $\hat{X}^* \in \mathcal{X}_v^*$. The optimal value of the objective function $E\hat{X}^*$ equals $\sum_{C \in 2^N} v(C)$ if and only if v is generated by X^*.*

Proof.

($\Rightarrow$) Assume that $E\hat{X}^* = \sum_{C\in 2^N} v(C)$. It follows from the feasibility constraints that for every $C \in 2^N \setminus \{\emptyset\}$

$$A_C(C)\hat{X}^* = v(C), \tag{23}$$

and $A_D(C)\hat{X}^* \geq v(C)$, for all $D \in 2^n$.

($\Leftarrow$) Assume that v is generated by X^* but $E\hat{X}^* \neq \sum_{C\in 2^N} v(C)$. From the feasibility constraints it follows that then

$$E\hat{X}^* > \sum_{C\in 2^N} v(C). \tag{24}$$

Note that $\min_D X^*_D(C)$ is attained in row C of X^*, otherwise we could decrease the objective function by substituting the row attaining the minimum for row C. Combining this with Equation (24) we obtain that there exists a coalition $C \in 2^N$ such that

$$\min_D X^*_D(C) > v(C), \tag{25}$$

which together with Equation (19) imply that v cannot be generated by X^*, a contradiction. □

Take any matrix of realization vectors $X \in \mathbb{R}^{z\times n}$, where z is a non-negative integer. Let $Y(X) \in \mathbb{R}^{(2^n-1)\times n}$ denote the matrix in which for all $C \in 2^N$ we have that $Y_{C\cdot}(X) = X_{B\cdot}$, where $B = \arg\min_D X_D(C)$. Then we have the following proposition characterizing all the matrices that generate a given totally balanced game.

Proposition 3.6. *Take any totally balanced game $(N, v) \in \Gamma_{tb}$. The matrix of realization vectors $X \in \mathbb{R}^{z\times n}$ generates v if and only if $\hat{Y}(X)$ is an optimal solution of (P_v), that is $\hat{Y}(X) \in \mathcal{X}^*_v$.*

Proof.

($\Rightarrow$) If X generates v then for all $C \subseteq N$ there exists a state of nature s_C such that

$$X_{s_C}(C) = v(C) \tag{26}$$

and for all $s \in \{1, \dots, S\}$ we have that

$$X_s(C) \geq v(C), \tag{27}$$

thus $\hat{Y}(X)$ is a feasible and optimal solution of (P_v).

($\Leftarrow$) If $\hat{Y}(X)$ is an optimal solution of (P_v) then by Proposition 3.5 v can be generated by $Y(X)$ and by construction also by X. □

Take any totally balanced game $(N, v) \in \Gamma_{\mathrm{tb}}$. We saw in Proposition 3.2 that v can be generated by a matrix of realization vectors, say X. Note that $X = Y(X)$ and by Proposition 3.6 $\hat{X}$ is an optimal solution of (P_v), $\hat{X} \in \mathcal{X}^*_v$.

We also have the following proposition about the core requirement (16).

Proposition 3.7. *Take any totally balanced game $(N, v) \in \Gamma_{tb}$. For any optimal solution of (P_v) $\hat{X}^* \in \mathcal{X}_v^*$ we have that X^*, the matrix derived from $\hat{X}^*$ satisfies the core requirement (16).*

Proof. Take any $\hat{X}^* \in \mathcal{X}_v^*$. For every $C \in 2^N \setminus \{\emptyset\}$

$$A_C(C)\hat{X}^* = v(C), \tag{28}$$

as the feasibility requires that $A_C(C)\hat{X}^* \geq v(C)$, and since by Proposition 3.2 all totally balanced games can be generated we know by Proposition 3.5 that $E\hat{X}^* = \sum_{C \in 2^N} v(C)$. The equalities in (28) together with the feasibility constraints imply that the rows of X^* contain core allocations of the respective subgames. □

Propositions 3.7 and 3.6 imply that if a game can be generated by $X \in \mathbb{R}^{z \times n}$ then $Y(X)$ satisfies the core requirement (16). Thus to generate a given totally balanced game the rows of the matrix of realization vectors can be permutated and some of them can be combined, but essentially the core requirement is satisfied in all of them.

4 Exactness

In this section we show that if there is no aggregate uncertainty in a risk environment then the induced risk allocation game is an exact game and all exact games can be generated by a properly specified risk environment with no aggregate uncertainty.

Proposition 4.1. *All risk allocation games with no aggregate uncertainty $v \in \Gamma_{rn}$ are exact, $\Gamma_{rn} \subseteq \Gamma_e$.*

Proof. Take any risk environment (n, S, p, X, ρ), where X is expressing no aggregate uncertainty. Using Theorem 2.7 we will show that the induced risk allocation game is exact, since it is totally balanced and overbalanced (Definition 2.8). Total balancedness was shown in Proposition 3.1 for a general risk allocation game. For overbalancedness take any collection of real numbers $(\lambda^C)_{C \subset N} \in \mathbb{R}_+$ such that $\sum_{C \in 2^N \setminus \{D,N\}} \lambda^C a(C) = a(N) + \lambda^D a(D)$. Then by Equation (3), the positive homogeneity and subadditivity of ρ we have that

$$\begin{aligned} \sum_{C \in 2^N \setminus \{D,N\}} \lambda^C v(C) &= - \sum_{C \in 2^N \setminus \{D,N\}} \rho(\sum_{i \in C} \lambda^C X_{\cdot i}) \leq -\rho(\sum_{C \in 2^N \setminus \{D,N\}} (\sum_{i \in C} \lambda^C X_{\cdot i})) = \\ &= -\rho(\sum_{i \in N} (\sum_{C \ni i, C \in 2^N \setminus \{D,N\}} \lambda^C X_{\cdot i})) = -\rho(\sum_{i \in N} X_{\cdot i} + \lambda^D \sum_{i \in D} X_{\cdot i}), \end{aligned} \tag{29}$$

where the last line follows from rearranging the summation and using the fact that we have an overbalanced collection of weights, thus if $i \in D$ then $\sum_{C \ni i, C \in 2^N \setminus \{D,N\}} \lambda^C = 1 + \lambda^D$

and if $i \notin D$ then $\sum_{C \ni i, C \in 2^N \setminus \{D,N\}} \lambda^C = 1$. Using translation invariance and positive homogeneity Equation (29) can be continued as

$$\sum_{C \in 2^N \setminus \{D,N\}} \lambda^C v(C) \leq -\rho(\sum_{i \in N} X_{\cdot i} + \lambda^D \sum_{i \in D} X_{\cdot i}) = -\rho(X(N)) - \rho(\lambda^D X(D)) =$$

$$= -\rho(X(N)) - \lambda^D \rho(X(D)) = v(N) + \lambda^D v(D), \tag{30}$$

thus we have an overbalanced game. □

Proposition 4.2. *For each exact game $(N, v) \in \Gamma_e$ there exist a risk environment with no aggregate uncertainty such that the induced risk allocation game equals (N, v), $\Gamma_e \subseteq \Gamma_{rn}$.*

Proof.

Take any exact game $(N, v) \in \Gamma_{\mathrm{e}}$. We specify the risk environment (n, S, p, X, ρ) inducing a game $(N, \bar{v})$ as follows. Let us introduce a state of nature for all proper subcoalitions of N, thus let $S = 2^n - 2$. Let $p_1 = \cdots = p_S = 1/S$, and let ρ be the 1-expected shortfall with $\delta = 1$. Since v is exact by definition for all $C \subset N$ there exist a core element x_C such that $x_C(C) = v(C)$. Construct $X \in \mathbb{R}^{S \times n}$ as follows. Labeling states of nature by the proper subcoalitions of N let for all $C \subset N$, $X_{C\cdot} = x_C$. Of course $X_{C\cdot}(N) = v(N)$, thus X is expressing no aggregate uncertainty. By construction we have that for every $C \in 2^N \setminus \{\emptyset\}$ the value function of the induced risk allocation game $(N, \bar{v})$ is given by

$$\bar{v}(C) = \min_{D \in 2^N \setminus \{\emptyset\}} X_D(C) = v(C), \tag{31}$$

thus $\bar{v} = v$. □

Note that in the proof of Proposition 4.2, $v(N)$ is set in all rows, that is why we need only $2^n - 2$ states of nature.

By combining Propositions 4.1 and 4.2 we have the following theorem.

Theorem 4.3. *The class of risk allocation games with no aggregate uncertainty coincides with the class of exact games, $\Gamma_{rn} = \Gamma_e$.*

Csóka, Herings, and Kóczy (2007a) show that if there are less than four players then the class of exact games coincides with the class of convex games. Using this result Theorem 4.3 can be reformulated as follows.

Theorem 4.4. *If there are less than four players then the class of risk allocation games with no aggregate uncertainty coincides with the class of convex games, $\Gamma_{rn} = \Gamma_c$.*

Theorem 4.4 is illustrated by the following example.

Example 4.5. In this example we show how a 3-player convex game can be generated by a risk allocation game with no aggregate uncertainty. Note that the game in Table

1 of Example 3.3 is not convex since $v(\{1,2\}) + v(\{1,3\}) = -4 - 6 = -10 > v(\{1\}) + v(\{1,2,3\}) = -10-1 = -11$. However, as can easily be verified, by changing $v(\{1,2\})$ to -5 we get the convex game shown in Table 3. This game is generated by the risk environment including the matrix of realization vectors X of Table 4 and the 1-expected shortfall with $\delta = 1$.

Notice that the rows of X correspond to appropriately chosen marginal contribution vectors. For instance, in the first row of X we have the marginal contributions corresponding to the permutation player 3, player 2, player 1: $v(\{3\}) - v(\{\emptyset\}) = -2 - 0 = -2; v(\{2,3\}) - v(\{2\}) = 2 - (-2) = 4; v(\{1,2,3\}) - v(\{2,3\}) = -1 - 2 = -3$. Since in a convex game the marginal contribution vectors are in the core, there are core elements distributing their value to n coalitions (the increasing subsets of players) at the same time. Thus to generate a convex game fewer states of nature are required, in this example only 3. Also note that all rows of X sum up to -1, since the sum of the marginal contributions is always the value of the grand coalition. Thus there is no aggregate uncertainty.

C	$v(C)$
$\{1\}$	-10
$\{2\}$	3
$\{3\}$	-2
$\{1,2\}$	-5
$\{1,3\}$	-6
$\{2,3\}$	2
$\{1,2,3\}$	-1

Table 3: The value function of a convex game v.

S	$X_{\cdot 1}$	$X_{\cdot 2}$	$X_{\cdot 3}$	$\sum X_{s,i}$
1	-3	4	-2	-1
2	-7	3	3	-1
3	-10	5	4	-1

Table 4: A matrix of realization vectors generating v.

Similarly to Proposition 3.6 we can characterize all the risk environments with the maximum loss that generate a given exact game.

Proposition 4.6. *Take any exact game $(N, v) \in \Gamma_e$. The matrix of realization vectors $X \in \mathbb{R}^{z \times n}$ satisfying no aggregate uncertainty generates v if and only if $\hat{Y}(X)$ is an optimal solution of (P_v), that is $\hat{Y}(X) \in \mathcal{X}_v^*$.*

Proof. Proposition 3.6 characterizes all the matrices that generate a given totally balanced game. Since by Proposition 4.1 only exact games can be generated with matrices satisfying no aggregate uncertainty, the proof is straightforward. □

5 Conclusion

In this paper we have discussed transferable utility cooperative games derived from a risk environment: risk allocation games. We have shown that the class of risk allocation games coincides with the class of totally balanced games. This result makes sure that a regulator or performance evaluator can always allocate risk in a stable way: there will always be a core element, no matter how the risk environment is changing.

We have also studied the case when the aggregate portfolio has the same payoff in all states of nature. We proved that if there is no aggregate uncertainty then the class of risk allocation games equals the class of exact games, where for each coalition there is a core element such that the coalition gets only its stand-alone value. This means that if there is no aggregate uncertainty, then not necessarily everybody benefits from the diversification effects in a stable allocation of risk. To put it differently, the lower aggregate uncertainty, the more power the regulator or performance evaluator has in allocating risk, since for each coalition there is always a stable allocation of risk, such that it gets hardly less than its stand-alone value.

We have characterized all the matrices of realization vectors that generate a given totally balanced or exact game. In both cases the vectors derived from the matrices by juxtaposing their rows are related to the optimal solutions of a linear programming problem.

Denault (2001) shows that if a risk allocation game for an arbitrary matrix of realization vectors is convex then the risk measure by which it is induced is necessarily additive, thus the generated risk allocation game is also additive. However, by imposing some structure on the matrix of realization vectors we have proven the following theorem: If there are less than four players then the class of convex games coincides with the class of risk allocation games having the discounted maximum loss, at most three states of nature and no aggregate uncertainty in their risk environments.

References

Acerbi, C., 2002. Spectral measures of risk: A coherent representation of subjective risk aversion. Journal of Banking and Finance 26, 1505–1518.

Acerbi, C., Tasche, D., 2002. On the coherence of expected shortfall. Journal of Banking and Finance 26, 1487–1504.

Artzner, Ph. F., Delbaen, F., Eber, J.-M., Heath, D., 1999. Coherent measures of risk. Mathematical Finance 9, 203–228.

Bondareva, Olga N., 1963. Some applications of linear programming methods to the theory of cooperative games (in Russian). Problemy Kybernetiki 10, 119–139.

Csóka, P., Herings, P. J. J., Kóczy, L. Á., 2007a. Balancedness conditions for exact games. Manuscript, pp. 1–14.

Csóka, P., Herings, P. J. J., Kóczy, L. Á., 2007b. Coherent measures of risk from a general equilibrium perspective. Journal of Banking and Finance, forthcoming, doi:10.1016/j.jbank.n.2006.10.026

Denault, M., 2001. Coherent allocation of risk capital. Journal of Risk 4, 1–34.

Dubey, P., Shapley, Lloyd S., 1982. Totally balanced games arising from controlled programming problems. Discussion paper.

Kalai, E., Zemel, E., 1982a. Generalized network problems yielding totally balanced games. Operations Research 30, 998–1008.

Kalai, E., Zemel, E., 1982b. Totally balanced games and games of flow. Mathematics of Operations Research 7, 476–478.

Legut, J., 1990. On totally balanced games arising from cooperation in fair division. Games and Economic Behavior 2, 47–60.

Owen, G., 1975. On the core of linear production games. Mathematical Programming 9, 358–370.

Predtetchinski, A., Herings, P. J. J., 2004. A necessary and sufficient condition for the non-emptiness of the core of a non-transferable utility game. Journal of Economic Theory 116, 84–92.

Schmeidler, D., 1972. Cores of exact games. Journal of Mathematical Analysis and Applications 40, 214–225.

Shapley, Lloyd S., 1967. On balanced sets and cores. Naval Research Logistics Quarterly 14, 453–460.

Shapley, L. S., Shubik, M., 1969. On market games. Journal of Economic Theory 1, 9–25.

Tijs, S., Parthasarathy, T., Potters, J., Prassad, V. Rajendra, 1984. Permutation games: Another class of totally balanced games. OR Spektrum 6, 119–123.