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Independence, Heterogeneity and Uniqueness in Interaction Games

by Robin Mason and Ákos Valentinyi

Abstract

This paper shows that incomplete information and sufficient heterogeneity of players can ensure uniqueness in interaction games. In contrast to recent work on uniqueness in interaction games, we do not require strategic complementarity. There are two parts to the argument. First, if a player's signal is sufficiently uninformative of the signals of its opponents (in the sense of the Fisher information of the signal), then the player's best response to any strategy profile of its opponents is non-decreasing in its signal. Secondly, a contraction mapping argument shows that sufficient heterogeneity ensures that equilibrium is unique.

Keywords: Co-ordination, Interaction games, Heterogeneity, Unique equilibrium

JEL Classification: C72; D82

Robin Mason – Valentinyi Ákos

Függetlenség, heterogenitás és az egyensúly egyértelműsége interakciós játékokban

Összefoglaló

A tanulmány megmutatja, hogy tökéletlen információ és megfelelő heterogenitás biztosítja az egyensúly egyértelműségét interakciós játékokban. Ellentétben az egyensúly egyértelműségére vonatkozó korábbi eredményekkel, mi nem tételezzük fel, hogy a játékosok akciói között stratégiai komplementaritás áll fenn.

Bizonyításunknak két része van. Először megmutatjuk, hogy ha egy játékos által kapott jelzés kellőképpen kevés információt tartalmaz a többi játékos által kapott jelzésről, akkor a játékos optimális stratégiája a kapott jelzés növekvő függvénye függetlenül attól, hogy ellenfelei milyen stratégiát játszanak. Másodszor, megmutatjuk, hogy ha az információ megfelelő, heterogenitása biztosítja az egyensúly egyértelműségét.

1 Introduction

Complete information interaction games (which include co-ordination, random matching and local interaction games—see Morris (1997)) often have multiple equilibria. In this paper, we show that when there is incomplete information, independence and sufficient heterogeneity of players can ensure uniqueness in this class of games. For any bounded interaction between players, we derive conditions under which a player's best response to any strategy profile of its opponents is a non-decreasing strategy or 'threshold strategy';[1] and under which there is a unique equilibrium in threshold strategies. Both parts require that

1. each player's payoff is sufficiently sensitive to its own signal;
2. each player's payoff is sufficiently insensitive to the actions of other player;
3. each player's signal is sufficiently uninformative about the signals of other players; and
4. the conditional probability of any signal is sufficiently small.

In the case of the normal distribution, the condition can be expressed simply in terms of the correlation ρ and variance σ of the signal distribution: for any sufficiently large σ, there is a critical ρ^*, strictly between 0 and 1, such that if the degree of correlation is less than ρ^*, then any equilibrium must be in threshold strategies and there is a unique threshold strategy equilibrium. The result can be extended to more general distributions. Then what matters is the *Fisher information*—a measure of how sensitive the likelihood of other players' signals is to the signal of an individual player—and the *conditional density.* The sufficient condition for uniqueness is that the Fisher information is bounded (i.e., a player's signal tells it little about the signals of other players); and that the conditional density also is bounded.

Previous work has shown in a variety of situations that heterogeneity can help to ensure uniqueness of equilibrium. For example, in a canonical two-by-two public good

[1] A threshold strategy in a binary action game, for example, specifies that one action is taken only for signals above some cutoff point.

model in Fudenberg and Tirole (1991, pp. 211–213), there are two pure strategy equilibria in the common knowledge game. There is only one equilibrium in the incomplete information game if the distribution of types satisfies certain conditions. One such condition is that the maximum value of the density is sufficiently small; following Grandmont (1992), this can be interpreted as requiring a sufficient degree of heterogeneity between the players. Herrendorf et al. (2000) show how heterogeneity in the manufacturing productivity (rather than the information) of agents in the two-sector, increasing returns-to-scale Matsuyama (1991) model can remove indeterminacy and multiplicity of equilibrium. Burdzy et al. (2000) show in a dynamic game that if agents are heterogeneous (in the sense of being unable to adjust behaviour at identical times), then exogenous shocks can lead to a unique equilibrium in the Matsuyama setting. Glaeser and Scheinkman (2002) show that if there is not too much heterogeneity among players, then there can be multiple equilibria in social interaction games.

The global games literature (see Carlsson and van Damme (1993), Morris and Shin (1998), and Morris and Shin (2002b)) also provides sufficient conditions for equilibrium uniqueness. These papers require that the players' actions are strict strategic complements i.e., a player's incentive to choose an action is increasing in the proportion of other players who choose that action.[2] Equilibrium uniqueness in a global game also requires sufficiently *small* heterogeneity—the players' signals must be sufficiently informative about the true underlying state, and hence highly correlated.

In this paper, we identify clearly the mechanism that is at work to establish uniqueness when there is a large degree of heterogeneity. We are able to contrast the mechanism with iterated deletion of dominated strategies based on higher-order beliefs that operates in the global game models.[3] With higher-order beliefs, sufficient (but not perfect) correlation between signals/types is required. Our mechanism relies on sufficient *lack of correlation* between signals. With independence of players' signals, the best response for any player

[2]Strictly speaking, Carlsson and van Damme do not require strategic complementarity. But, as Morris and Shin (2002a) note, when a two player, two action game has multiple Nash equilibria (the interesting case for Carlsson and van Damme's analysis), there are automatically strategic complementarities.

[3]Global games are supermodular, so that payoffs are strictly increasing in the mass of agents taking the same action. It is this structure that allows iterated deletion of dominated strategies; see Milgrom and Roberts (1990).

to any strategy by all other players is a threshold strategy. It is then straightforward to establish that there is a unique equilibrium in threshold strategies if and only if there is sufficient heterogeneity of signals. The analysis therefore resolves an open question concerning how different forms of heterogeneity can ensure uniqueness.

Our results are also of applied relevance. There is an increasing number of applications of the global game framework. Morris and Shin (1998) use the idea to analyze currency attacks and Morris and Shin (forthcoming) use it to analyze the pricing of debt. Karp (1999) applies it to Krugman (1991)'s two-sector model. In all these papers, strategic complementarity is assumed. But there are many applications in which this assumption is inappropriate, and where it would (for the usual reasons) be very useful to have a unique equilibrium. For example, in industrial organization, it is reasonable that positive network effects might hold in a new market when a small number of firms have entered; but that the network effects become negative once too many firms enter and the market becomes crowded. In the Internet, each new web site, or the addition of information to an existing site, increases the value of the Internet to every existing user. However, as usage of the Internet grows, so does congestion. Goldstein and Pauzner (2002) study a model of bank runs based on Diamond and Dybvig (1983). In their model, an agent's incentive for early withdrawal of funds from a bank is non-monotonic in the number of agents withdrawing. The incentive is highest when the number of agents demanding withdrawal reaches the level at which the bank goes bankrupt; after that point, the incentive decreases. (Despite this lack of complete strategic complementarity, Goldstein and Pauzner are able to establish uniqueness of equilibrium.) We do not require global strategic complementarities, only bounded interactions; hence our results can be used in a wider range of applications.

The rest of the paper is structured as follows. In section 2, we analyze a simple model, based on a particular payoff function and the normal distribution, to make the basic points of the paper. We extend the analysis in section 3 to show how the conclusions can be generalized to other distributions and payoffs. Section 4 concludes. Longer proofs are in the appendix.

2 A Simple Model

Suppose that there is a continuum of players, of measure 1. There are two possible actions. The payoff to any player from action 0 is zero. The payoff to player i from action 1 is $\gamma\hat{x} + (1-\gamma)x_i + f(n)$. x_i is player i's private signal, observed only by player i. It is drawn from a normal distribution with mean y and variance σ^2. Players' signals are correlated—the degree of correlation between the signals of player i and $j \neq i$ is $\rho \in [0,1)$ (note that perfect correlation is ruled out). Hence when player i has a private signal of x_i, its posterior of the signal x_{-i} of any player $-i$ is normally distributed with mean $\rho x_i + (1-\rho)y$ and variance $\sigma^2(1-\rho^2)$. y, σ^2 and ρ are common knowledge. $\hat{x}$ is the mean signal of all players; unconditionally, it is equal to y. The parameter γ lies in the interval $[0,1]$; if it is equal to 0, the model is one of private values; if it is 1, it is a pure common value model; for intermediate values of γ, there is a limited degree of common value. Finally, $n \in [0,1]$ is the proportion of players choosing action 1. $f : [0,1] \to \mathbb{R}$ is the interaction function, describing how a player's utility is affected by the actions of other players. We assume that it is continuous and bounded i.e., there exists a finite κ such that $\sup_{n\in[0,1]} |f(n)| \leq \kappa/2$. The assumption implies that there are dominance regions: any rationalizable strategy must involve playing 0 for any valuation less than $\underline{x} \equiv -\sup f$, and playing 1 for any valuation greater than $\bar{x} \equiv -\inf f$.

Consider any strategy profile played by all players other than i. This profile induces a distribution $s(x) : \mathbb{R} \to [0,1]$ that gives the proportion of players choosing action 1 for a given value of x. The expected utility gain for player i of choosing action 1, conditional on receiving the signal x_i, is then

$$\begin{aligned} \mathbb{E}[\Delta u(x_i, s)] &\equiv \gamma\left(\rho x_i + (1-\rho)y\right) + (1-\gamma)x_i \\ &\quad + \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} f(s(x)) \exp\left[-\frac{1}{2}\left(\frac{x - \rho x_i - (1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)^2\right] dx. \quad (1) \end{aligned}$$

So player i's expected utility has two components: the expected stand-alone utility (the first line of the expression), and the expected interaction utility (the second line).

2.1 The Independent Private Value Case

Consider first the case of independent signals ($\rho = 0$) and private values ($\gamma = 0$). Clearly in this case, the expected interaction utility does not depend on player i's signal. It is then straightforward that $\mathbb{E}[\Delta u(x_i, s)]$ is a strictly increasing function of x_i for any $s(\cdot)$. And, because of the dominance regions, this means that the best response to any distribution $s(\cdot)$ induced by any strategy profile is a threshold strategy.

Proposition 1 *In the independent private value case, $\gamma = \rho = 0$, the best response $BR(s)$ to any distribution $s(\cdot)$ induced by any strategy profile is a threshold strategy i.e., takes the form*

$$BR(s) = \begin{cases} 0 & x < \tilde{x}, \\ 1 & x \geq \tilde{x} \end{cases}$$

for some $\tilde{x} \in (\underline{x}, \bar{x})$.

Hence, any equilibrium must be in threshold strategies. Given the threshold point $\tilde{x}$ in a symmetric threshold strategy equilibrium, the expected utility of a player who receives a signal $\tilde{x}$ is

$$\mathbb{E}[\Delta u(\tilde{x})] \equiv \tilde{x} + \frac{1}{\sqrt{2\pi}\sigma} \left(\int_{-\infty}^{\tilde{x}} f(0) \exp\left[-\frac{1}{2} \left(\frac{x-y}{\sigma} \right)^2 \right] dx + \int_{\tilde{x}}^{\infty} f(1) \exp\left[-\frac{1}{2} \left(\frac{x-y}{\sigma} \right)^2 \right] dx \right). \quad (2)$$

The equilibrium threshold point satisfies the equation

$$\mathbb{E}[\Delta u(\tilde{x})] = 0. \quad (3)$$

MS show in the case of strict strategic complements (i.e., $f(\cdot)$ strictly increasing) that a necessary and sufficient condition for there to be a unique solution to equation (3) is that σ is sufficiently large i.e., that there is enough heterogeneity. A similar argument is given in HVW, who give a sufficient, but not necessary condition based on heterogeneity. The next proposition shows that the assumption of strategic complementarity is not needed for this result.

Proposition 2 *For any continuous and bounded interaction function $f(\cdot)$, in the independent private value case, there exists a $\sigma^* \geq 0$ such that if $\sigma > \sigma^*$, then there is a unique equilibrium .*

Proof. There is a unique rationalizable action for (almost) all signals iff $d\mathbb{E}[\Delta u(\tilde{x})]/d\tilde{x} > 0$ for any $\tilde{x}$ at which $\mathbb{E}[\Delta u(\tilde{x})] = 0$. Differentiation of equation (2) shows that

$$\frac{d\mathbb{E}[\Delta u(\tilde{x})]}{d\tilde{x}} = 1 + \left(\frac{f(0) - f(1))}{\sqrt{2\pi}\sigma}\right) \exp\left[-\left(\frac{\tilde{x} - y}{\sigma}\right)^2\right].$$

Since $|f(0) - f(1)| \leq \kappa$, a sufficient condition for $d\mathbb{E}[\Delta u(\tilde{x})]/d\tilde{x} > 0$ is

$$1 > \frac{\kappa}{\sqrt{2\pi}\sigma}$$

which completes the proof. □

2.2 Positive Correlation and Common Values

Now suppose that there is a degree of correlation: $\rho \in (0, 1)$, and of common values: $\gamma \in (0, 1]$. In this section, we derive joint conditions on heterogeneity σ, correlation ρ, the common value parameter γ and the interaction function bound κ such that the best response of player i to any strategy profile played by all other players is a threshold strategy. Once this fact is established, sufficient heterogeneity again ensures uniqueness of equilibrium. Hence the basic mechanism that generates uniqueness in the case of independence extends to positive, but limited correlation, and to common values.

Proposition 3 *If*

$$(1 - \gamma(1 - \rho))\sqrt{\frac{1 - \rho^2}{\rho^2}} > \frac{\kappa}{\sqrt{2\pi}\sigma}, \tag{4}$$

then the best response to any strategy profile is a threshold strategy.

(The proposition follows from straightforward and lengthy algebra and so the proof is relegated to the appendix.)

In order to establish uniqueness of equilibrium in the correlated, common value case, we now derive a condition for there to be a unique threshold strategy equilibrium, assuming that such an equilibrium exists. This result is stated in proposition 4; as in proposition 2, it basically requires sufficiently large heterogeneity (for any given values of ρ and κ). We then combine the results of propositions 3 and 4 to give a sufficient condition for equilibrium uniqueness.

Proposition 4 *If*

$$(1-\gamma(1-\rho))\sqrt{\frac{1+\rho}{1-\rho}} > \frac{\kappa}{\sqrt{2\pi}\sigma}, \tag{5}$$

and a threshold strategy equilibrium exists, then there is a unique threshold strategy equilibrium.

Proof. As in the proof of proposition 2, there is a unique threshold strategy action for (almost) all signals iff $d\mathbb{E}[\Delta u(\tilde{x})]/d\tilde{x} > 0$ for any $\tilde{x}$ at which $\mathbb{E}[\Delta u(\tilde{x})] = 0$, where

$$\begin{aligned}
\mathbb{E}[\Delta u(\tilde{x})] \equiv\ & \gamma\left(\rho\tilde{x} + (1-\rho)y\right) + (1-\gamma)\tilde{x} \\
& + \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}}\left(\int_{-\infty}^{\tilde{x}} f(0)\exp\left[-\frac{1}{2}\left(\frac{x-\rho\tilde{x}-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)^2\right]dx\right. \\
& \left. + \int_{\tilde{x}}^{\infty} f(1)\exp\left[-\frac{1}{2}\left(\frac{x-\rho\tilde{x}-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)^2\right]dx\right).
\end{aligned}$$

Differentiation shows that a sufficient condition for $d\mathbb{E}[\Delta u(\tilde{x})]/d\tilde{x} > 0$ is

$$1-\gamma(1-\rho) > \frac{\kappa}{\sqrt{2\pi}\sigma}\left(\frac{1-\rho}{\sqrt{1-\rho^2}}\right).$$

This completes the proof. □

Proposition 5 *If*

$$(1-\gamma(1-\rho))\min\left[\sqrt{\frac{1-\rho^2}{\rho^2}},\sqrt{\frac{1+\rho}{1-\rho}}\right]>\frac{\kappa}{\sqrt{2\pi}\sigma} \tag{6}$$

then there is a unique equilibrium (which is in threshold strategies).

Proof. To have a unique equilibrium in threshold strategies, equations (4) and (5) must both hold. Also observe that

$$\sqrt{\frac{1-\rho^2}{\rho^2}} \geq \sqrt{\frac{1+\rho}{1-\rho}} \quad \text{for } \rho \in [0,\tfrac{1}{2}]$$
$$\sqrt{\frac{1-\rho^2}{\rho^2}} \leq \sqrt{\frac{1+\rho}{1-\rho}} \quad \text{for } \rho \in [\tfrac{1}{2},1).$$

So condition (5) implies (4) for $\rho \in (0,\frac{1}{2}]$ while the converse holds for $\rho \in [\frac{1}{2},1)$. The result follows. □

Proposition 5 gives a joint condition on the model parameters ρ, σ, γ and κ that is sufficient for equilibrium uniqueness. The proposition is illustrated in figures 1 and 2, which give an intuitive interpretation of the result.

Three facts stand out from the figures. First, our sufficient condition for uniqueness of equilibrium is stricter than that of MS. In figure 1, for example, the MS result gives a unique equilibrium for all parameter values lying in the area under the upward-sloping curve. We require in addition that parameter values lie in the area beneath the downward-sloping line. But, in contrast to MS, we do not require that players' actions are strategic complements—proposition 5 holds for any bounded interactions between the players. So, while our sufficient condition is indeed stricter than MS's when actions are strategic complements, it is less strict in the sense that it applies to a larger class of games.

Secondly, the figures demonstrate the statements made in the introduction of the paper—that there is a unique equilibrium (in threshold strategies) if and only if there is sufficient heterogeneity of signals. In figure 1, the sufficient condition requires the correlation between players' signals to be sufficiently low (and/or the variance of the

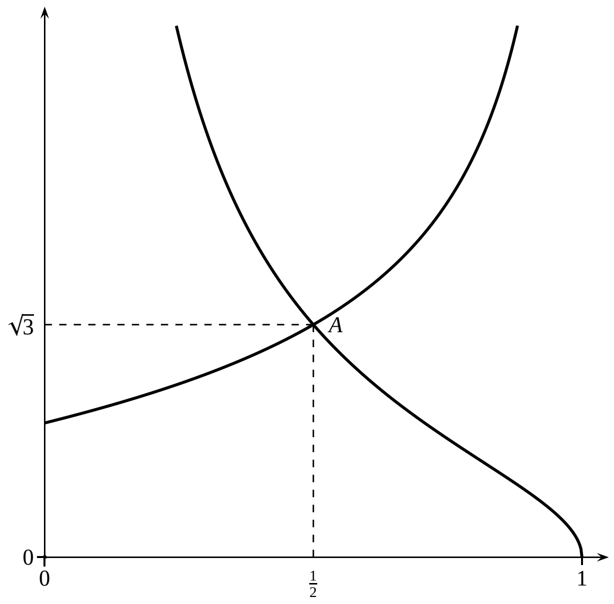

Figure 1: Proposition 5 for the Pure Private Value Case

prior distribution sufficiently high). (For certain parameter values, there is also a lower bound on the value of ρ.) In contrast, the MS condition alone generally requires the degree of correlation to be sufficiently *high*.

Finally, comparison of the two figures shows that allowing for common values decreases the parameter space over which the sufficient condition in proposition 5 holds. But the qualitative features of the result are unchanged by the presence of a common value component in players' payoffs. In this sense, the conclusion from the independent private value case carries over to the correlated common value case. In all cases, strategic complementarity is not required. This fact highlights that the mechanism at work here—the conditions ensure that a threshold strategy is a best response to all other strategies; and that there is a unique threshold strategy equilibrium—is quite different from the mechanism analyzed by MS.

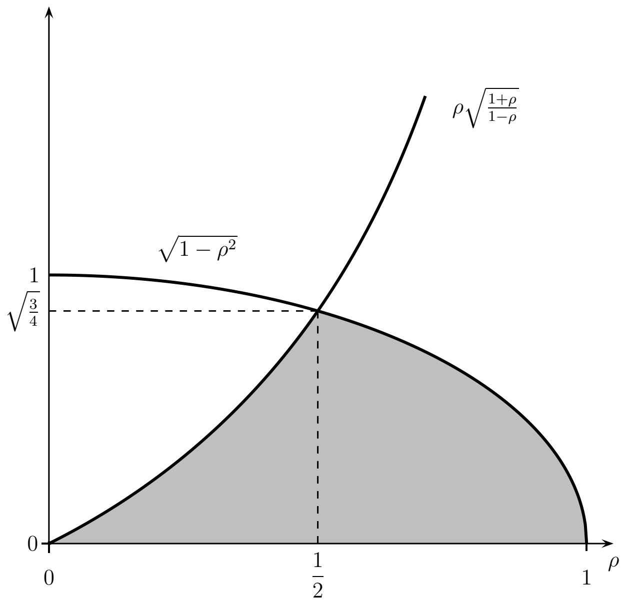

Figure 2: Proposition 5 for the Pure Common Value Case

3 The General Model

The simple model establishes the role that independence, and hence small correlation, plays in ensuring uniqueness in the interaction game. There is a possibility, however, that the conclusions depend on the simplifying assumptions of the model. In this section, we extend the model in a few directions to show that this is not the case. In particular, we allow for a more general payoff structure and distribution of signals.

3.1 The Private Value, Countable Action Case

Consider a game of incomplete information between a continuum of players of measure 1 where each player observes first observes her own information, and then chooses an action. Each player observes a signal x_i drawn from a common distribution $G(x)$ with support X. Following Frankel et al. (2003), partition the set of players into a finite set T of "types" of players. Each type contains either a single player or a continuum of players. Moreover, each type is of finite measure (normalized so that the total population is of measure 1). For each player $i \in [0,1]$ let $t_i \in T$ be the type of i, and T_{-i} the set of types of player i's opponents.

The action set A_i of each player i is a closed, finite subset of the unit interval that contains 0 and 1 i.e., $\{0,1\} \subseteq A_i \subset [0,1]$. Player i's payoff from choosing action $a_i \in A_i$ on receiving a signal $x_i \in X$ and being of type $t_i \in T$ is $u_i(a_i, x_i, t_i, \boldsymbol{\alpha}_{-i})$ where α_t is the cumulative distribution function (cdf) of actions chosen by type-t players (that is $\alpha_t(a)$ is the proportion of type t players who play action a or less), and $\boldsymbol{\alpha}_{-i} = (\alpha_t)_{t \in T_{-i}}$ is the vector of cdfs of player i's opponents. Note that players of the same type may have different payoff functions and may have different action sets. Note also that the game is not truly anonymous. Within a type, players are anonymous; but a player's identity matters through its type.

Let

$$\Delta u_i(a_i, a_i', x_i, t_i, \boldsymbol{\alpha}_{-i}) \equiv u_i(a_i, x_i, t_i, \boldsymbol{\alpha}_{-i}) - u_i(a_i', x_i, t_i, \boldsymbol{\alpha}_{-i}).$$

We make the following assumptions on payoff functions.

U1. Limit Dominance. There exist $\underline{x}_i$ and $\bar{x}_i$ such that

(a) $\Delta u_i(0, a_i', x_i, t_i, \boldsymbol{\alpha}_{-i}) > 0$ for all $a_i' \neq 0$, $t_i \in T$, $\boldsymbol{\alpha}_{-i}$, and $x_i \leq \underline{x}_i$,

(b) $\Delta u_i(1, a_i', x_i, t_i, \boldsymbol{\alpha}_{-i}) > 0$ for all $a_i' \neq 1$, $t_i \in T$, $\boldsymbol{\alpha}_{-i}$, and $x_i \geq \bar{x}_i$.

Let $\underline{x} = \inf_{i \in [0,1]} \underline{x}_i$ and $\bar{x} = \sup_{i \in [0,1]} \bar{x}_i$.

U2. Uniformly Positive Sensitivity to State. There is a $\delta \in (0, \infty)$ such that for all $a_i \geq a_i'$, $x_i \geq x_i'$, i and t_i

$$\Delta u_i(a_i, a_i', x_i, t_i, \boldsymbol{\alpha}_{-i}) - \Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\alpha}_{-i}) \geq \delta(a_i - a_i')(x_i - x_i').$$

U3. Uniformly Bounded Sensitivity to Opponents' Action. There is a $\kappa \in (0, \infty)$ such that

$$\Delta u_i(a_i, a_i', x_i, t_i, \boldsymbol{\alpha}_{-i}) - \Delta u_i(a_i, a_i', x_i, t_i, \boldsymbol{\alpha}_{-i}') \leq \kappa(a_i - a_i')|\boldsymbol{\alpha}_{-i} - \boldsymbol{\alpha}_{-i}'|$$

where

$$|\boldsymbol{\alpha}_{-i} - \boldsymbol{\alpha}'_{-i}| = \sup_{t \in T_{-i}} \sup_{a \in A_t} |\alpha_t(a) - \alpha'_t(a)|$$

and $A_t = \cup_{j|t_j=t} A_j$, for all i and t_i.

We make the following assumptions about the players' signals.

D1. $G(x)$ is atomless and its support X includes $[\underline{x}, \bar{x}]$. The density $g(x)$ is bounded. The conditional density $g(x|x_i)$ is differentiable with respect to x_i.

D2. There is a $\iota \in (0, \infty)$ such that $\iota \geq \max_{x_i \in X} \sqrt{I(x_i)}$ where

$$I(x_i) = \text{Var}\left(\frac{\partial \ln g(x|x_i)}{\partial x_i}\right)$$

is the Fisher information in x_i about the signal of the opponents.

D3. For all $x, x_i \in X$, there is a $\eta \in [0, +\infty)$ such that $g(x|x_i) \leq \eta$.

Players use distributional strategies. A distributional strategy for player i is a probability measure on $A_i \times X \times T$ such that the marginal distribution on X is $g(x)$; see Milgrom and Weber (1985). Let $\mu_i(x_i, t_i)$ be the cumulative distribution function of player i of type t_i who receives a signal x_i i.e., $\mu_i(a, x_i, t_i)$ is the probability that player i plays $a_i \leq a$ if he receives signal x_i and is of type t_i. Let $\mu_t(a, x) = \int_{j|t_j=t} \mu_j(a, x, t_j) dj$ be the cdf played by type t players, $\boldsymbol{\mu}_{-i}(a, x) = (\mu_t(a, x))_{t \in T_{-i}}$ be the corresponding vector of opponents' cdfs, and $\boldsymbol{\mu}(a, x) = (\mu_t(a, x))_{t \in T}$ be the vector of cdfs of all types. The expected payoff to player i is given by

$$\mathbb{E}[u_i(a_i, x_i, t_i, \boldsymbol{\mu}_{-i})] = \int_X u_i(a_i, x_i, t_i, \boldsymbol{\mu}_{-i}(a, x)) g(x|x_i) dx.$$

In the next lemma (the proof of which is in the appendix), we derive a sufficient condition that ensures that a player's expected payoff function satisfies the strict single crossing condition. We then use this property in proposition 6 to argue that all players use threshold strategies i.e., strategies that are non-decreasing in their signal.

Lemma 1 *If assumptions U1–U3 and D1–D2 hold and* $\iota < \delta/\kappa$*, then player* i*'s expected payoff satisfies the strict single crossing property in* (a_i, x_i) *for any* $t_i \in T$*, and* $\boldsymbol{\mu}_{-i}$ *i.e.,* $\mathbb{E}[u_i(a_i, x_i', t_i, \boldsymbol{\mu}_{-i})] \geq \mathbb{E}[u_i(a_i', x_i', t_i, \boldsymbol{\mu}_{-i})]$ *implies* $\mathbb{E}[u_i(a_i, x_i, t_i, \boldsymbol{\mu}_{-i})] > \mathbb{E}[u_i(a_i', x_i, t_i, \boldsymbol{\mu}_{-i})]$ *for any* $a_i, a_i' \in A_i$ *and for all* $x_i > x_i'$.

Proposition 6 *If assumptions U1–U3 and D1–D2 hold and* $\iota < \delta/\kappa$*, then the best response of player* i *to any profile of opponents' strategies is monotone non-decreasing in her signal* x_i.

Proof. The action set A_i is totally ordered because $\{0, 1\} \subseteq A_i \subset [0, 1]$ implying that $\mathbb{E}[u_i(a_i, x_i, t_i, \boldsymbol{\mu}_{-i})]$ is quasisupermodular in a_i. Moreover, A_i is independent of x_i, and $X \in \mathbb{R}$ is also totally ordered. Finally, $\mathbb{E}[u_i(a_i, x_i, t_i, \boldsymbol{\mu}_{-i})]$ satisfies the strict single crossing property when $\iota < \delta/\kappa$, from lemma 1. Therefore by the Monotone Selection Theorem 4' of Milgrom and Shannon (1990),

$$s_i^*(x_i, t_i, \boldsymbol{\mu}_{-i}) = \arg\max_{a_i \in A_i} \mathbb{E}[u_i(a_i, x_i, t_i, \boldsymbol{\mu}_{-i})]$$

is monotone non-decreasing in x_i. □

The sufficient condition in proposition 6 that ensures that each agent plays a threshold strategy is stronger than that found in the simple model of section 2 (see proposition 2). The Fisher information with the normal distribution is

$$I(x_i) = \frac{\rho^2}{\sigma^2(1 - \rho^2)};$$

in contrast, the sufficient condition in proposition 4 bounds

$$\frac{\rho^2}{2\pi\sigma^2(1 - \rho^2)}.$$

The factor of 2π that does not appear in the bound in this section means that the sufficient condition in proposition 6 is more demanding. Nevertheless, it is doing much the same work as the condition in proposition 4. Both require that a player's signal tells it sufficiently little about the signals of other players—in the case of proposition 4, by ensuring that heterogeneity is sufficiently large and/or correlation sufficiently small; in the case of proposition 6, by bounding the Fisher information.

The second step is to show that there is a unique equilibrium in threshold strategies. The direct argument for this step used in the simple model of the previous section cannot be applied in this more general model. MS and Frankel et al. (2003) show that the extension of this argument to a more general model with (potentially) asymmetric players requires the assumption of strategic complementarity.[4] In order to generalize beyond strict strategic complementarity, we use an argument that establishes that the mapping defining the equilibrium distribution is a contraction under a particular metric.

Proposition 7 *If assumptions U1–U3 and D1–D3 hold and*

$$\iota + \frac{\eta}{\lambda} \leq \frac{\delta}{\kappa} \tag{7}$$

where $\lambda < 1$*, then there is a unique equilibrium (which is in threshold strategies).*

The proof of the proposition, contained in the appendix, is technical and long. But the end result is simple, requiring only that ι be less than $\delta/\kappa - \eta/\lambda$. Note that compared to proposition 6, which requires only that ι is less than δ/κ, the sufficient condition in proposition 7 is stricter.

What is condition (7) ensuring? It does the two things that were illustrated in the simple model in section 2. First, it ensures that a player's own signal dominates interaction effects in payoff terms enough to make any best response a non-decreasing pure strategy. Roughly speaking, if condition (7) is satisfied, then each player places more weight on its own signal than on the possible actions of its opponents when choosing its best action. Secondly, the condition ensures that there is a unique equilibrium in non-decreasing pure strategies. It does so by showing in the general case the mechanism that was shown for the

[4] By the results of Milgrom and Roberts (1990) on supermodular games, there exists a largest and a smallest Nash equilibrium. Suppose that these are distinct equilibria. In a global game in which actions are strategic complements, players' payoffs are increasing in the level of the state, all other things equal, and private signals of the state are sufficiently precise, this leads to a contradiction.

binary action case. In order for there to be multiple equilibria in non-decreasing strategies, it must be that there are multiple values of a player's signal that leaves that player indifferent between the two actions. The direct effect of a player's signal is monotonic: the utility difference between the actions increases with the signal, other things equal. So, in order for there to be multiple equilibria, the indirect effect, operating through the player's assessment of its opponents' actions, must dominate. Condition (7) ensures that the direct, own-signal effect is sufficiently strong; or that the interaction effect is sufficiently weak; or that the player's signal is sufficiently uninformative about the information (and hence likely action) of others. It therefore ensures that the direct effect dominates and multiplicity is not possible.

As a final note, it is worth comparing condition (7) with condition (5) established in proposition 4. Recall that there, a contraction mapping was found for non-decreasing pure strategies when

$$\frac{\delta}{\kappa} > \frac{1}{\sigma\sqrt{2\pi}}\sqrt{\frac{1-\rho}{1+\rho}}$$

in the private value case. (In fact, $\delta = 1$ in the simple model; it is written here as a general parameter for comparability.) Condition (7) requires that

$$\frac{\delta}{\kappa} \geq \frac{1}{\sigma}\frac{\rho}{\sqrt{1-\rho^2}} + \frac{1}{\lambda\sigma\sqrt{2\pi}\sqrt{1-\rho^2}}$$

where the expressions for the Fisher information and the maximum value of the density of the normal have been used. Condition (7) therefore implies condition (5) if

$$\frac{1}{\sigma}\frac{\rho}{\sqrt{1-\rho^2}} + \frac{1}{\lambda\sigma\sqrt{2\pi}\sqrt{1-\rho^2}} > \frac{1}{\sigma\sqrt{2\pi}}\left(\sqrt{\frac{1-\rho}{1+\rho}}\right)$$

i.e., $\rho(1+\sqrt{2\pi}) > 1 - 1/\lambda$, which certainly holds since $\lambda < 1$. In summary: the sufficient condition in proposition 7 is stricter than the sufficient condition in proposition 4.

3.2 Extensions

The previous section established a sufficient condition for equilibrium uniqueness for the case of private values and a finite action set for each player. In this section, we show that extending the result beyond this case is straightforward.

To allow for interdependent values, now suppose the utility of player i of type $t_i \in T$ receiving signal $x_i \in X$ from choosing action $a \in A$ is $u_i(a, x_i, t_i, \theta, \boldsymbol{\alpha}_{-i})$ where $\theta \in X^m$ is an unobserved statistic of all (other) players' signals. In order to avoid dealing with infinite-dimensional integrals, we assume that m is finite. In the private values case, u_i is not a function of θ for all i. Let $h(\theta|x)$ denote a player's probability density function for the statistic θ conditional on receiving a signal x, with support X. Note that the conditional density $h(\cdot)$ is symmetric across players.

In line with the previous approach, we make the following additional assumptions:

U4. Bounded variation: For any actions $a_i, a_i' \in A_i$, $x_i \in X$, $t_i \in T$ and vector of opponents' cdfs $\boldsymbol{\alpha}_{-i}$, there exists a $\omega \in (0, +\infty)$ such that $\mathrm{Var}_\theta\left[\Delta u_t(a_i, a_i', x_i, t_i, \theta, \boldsymbol{\alpha}_{-i})\right] \leq \omega^2$ where the variance is defined in terms of the conditional density $h(\theta|x_i)$.[5]

D4. For all $x_i \in X$ and $\theta \in X^m$, there is a $\nu \in [0, +\infty)$ such that $h(\theta|x_i) \leq \nu$.

With these two additional assumptions (and the previous assumptions adapted in an obvious way), the arguments in the previous section can be applied to give equivalent results. In particular,

Proposition 8 *If assumptions U1–U4 and D1–D4 hold and*

$$\iota + \frac{\eta\nu}{\lambda} \leq \frac{\delta}{\kappa\omega} \tag{8}$$

where $\lambda < 1$, then there is a unique equilibrium in the interdependent valuation case.

[5] That is,

$$\mathrm{Var}_\theta\left[\Delta u_i(\cdot)\right] = \int_{X^m} \left(\Delta u_i(\cdot) - \int_{X^m} \Delta u_i(\cdot) h(\theta|x_i) d\theta\right)^2 h(\theta|x_i) d\theta.$$

(The proof of the proposition is very similar to the proof of proposition 7 and so is omitted.) Proposition 8 shows that allowing for interdependent valuations modifies the sufficient condition for equilibrium uniqueness in a simple way. In particular, the bounding parameter δ in the private value case can simply be replaced by the ratio of parameters, δ/ω, which measures the relative importance of private and interdependent valuation components.

Consider now the case where each player's action set is a continuum. The argument of Athey (2001) (theorem 2) can be used in a direct way to establish the uniqueness of equilibrium in this case. One more assumption is required:

U5. Payoff Continuity. Each $u_i(a_i, x_i, t_i, \boldsymbol{\alpha}_{-i})$ is continuous in a_i and $\boldsymbol{\alpha}_{-i}$.

Then

Proposition 9 *If assumptions U1–U5 and D1–D4 hold and*

$$\iota + \frac{\eta\nu}{\lambda} \leq \frac{\delta}{\kappa\omega} \tag{9}$$

where $\lambda < 1$*, then there is a unique equilibrium in the interdependent valuation case with a continuum of actions.*

The proposition follows immediately from Athey (2001) and proposition 8 (noting that any sequence that converges has a unique limit).

4 Conclusions

In this paper, we have provided a sufficient condition for there to be a unique equilibrium in interaction games. Our framework can be applied to a broad class of games; for example, we do not require the assumption of global strategic complementarity. We have therefore been able to clarify the mechanism that is at work when heterogeneity generates uniqueness. In addition to this theoretical contribution, our approach can be used in a number of applications in which externalities can be both positive and negative.

A Proof of Proposition 3

A sufficient condition for player i's best response to any distribution $s(\cdot)$ induced by any strategy profile to be a threshold strategy is that the expected utility $\Delta u(x_i, s)$ (see equation (1)) is an increasing function of x_i. This requires that

$$1-\gamma(1-\rho) \;>\; \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}}\left|\frac{\partial}{\partial x_i}\left(\int_{-\infty}^{+\infty} f(s(x))\exp\left[-\frac{1}{2}\left(\frac{x-\rho x_i-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)^2\right]dx\right)\right|$$
$$= \frac{\rho}{\sqrt{2\pi}\sigma^2(1-\rho^2)}\left|\int_{-\infty}^{+\infty} f(s(x))\left(\frac{x-\rho x_i-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)\exp\left[-\frac{1}{2}\left(\frac{x-\rho x_i-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)^2\right]dx\right|.$$

Since the normal distribution is symmetric around the mean,

$$\frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}}\left|\int_{-\infty}^{+\infty} f(s(x))\left(\frac{x-\rho x_i-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)\exp\left[-\frac{1}{2}\left(\frac{x-\rho x_i-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)^2\right]dx\right|$$
$$\le \frac{\kappa}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}}\int_{\rho x_i+(1-\rho)y}^{+\infty}\left(\frac{x-\rho x_i-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)\exp\left[-\frac{1}{2}\left(\frac{x-\rho x_i-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)^2\right]dx.$$

A change of variables

$$z \equiv \frac{1}{2}\left(\frac{x-\rho x_i-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)^2 \qquad dx \equiv \left(\frac{\sigma^2\sqrt{1-\rho^2}}{x-\rho x_i-(1-\rho)y}\right)dz$$

shows that

$$\frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}}\int_{\rho x_i+(1-\rho)y}^{+\infty}\left(\frac{x-\rho x_i-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)\exp\left[-\frac{1}{2}\left(\frac{x-\rho x_i-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)^2\right]dx$$
$$= \frac{\rho}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}}.$$

Hence the sufficient condition is

$$1-\gamma(1-\rho) > \frac{\kappa\rho}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}}$$

which proves the claim.

B Proof of Lemma 1

For the strict single crossing property to hold, it is sufficient to show that if $\iota < \delta/\kappa$, then

$$\mathbb{E}[\Delta u_i(a_i, a_i', x_i, t_i, \boldsymbol{\mu}_{-i})] > \mathbb{E}[\Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i})]$$

for $x_i > x_i'$. So we have to show that the difference between the expected values above is positive.

$$\begin{aligned}
&\mathbb{E}[\Delta u_i(a_i, a_i', x_i, t_i, \boldsymbol{\mu}_{-i})] - \mathbb{E}[\Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i})] \\
&\quad = \int_X \Delta u_i(a_i, a_i', x_i, t_i, \boldsymbol{\mu}_{-i}(x)) g(x|x_i) dx - \int_X \Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i}(x)) g(x|x_i') dx \\
&\quad = \int_X \left[\Delta u_i(a_i, a_i', x_i, t_i, \boldsymbol{\mu}_{-i}(x)) - \Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i}(x))\right] g(x|x_i) dx \\
&\qquad\quad - \int_X \Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i}(x)) \left[g(x|x_i') - g(x|x_i)\right] dx \\
&\quad \geq \delta |x_i - x_i'| - \int_X \Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i}(x)) \left[g(x|x_i') - g(x|x_i)\right] dx,
\end{aligned} \tag{10}$$

where the last inequality follows from assumption U2 if we also take into account that $a_i, a_i' \in [0, 1]$.

Let $\hat{x}_i \equiv \arg\max_{x_i \in X} [\partial g(x|x_i)]/[\partial x_i]$. Then the last term in equation (10) can be rearranged as

$$\begin{aligned}
&(x_i - x_i') \int_X \Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i}(x)) \frac{g(x|x_i') - g(x|x_i)}{x_i - x_i'} dx \\
&\quad \leq (x_i - x_i') \int_X \Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i}(x)) \left.\frac{\partial g(x|x_i)}{\partial x_i}\right|_{x_i = \hat{x}_i} dx \\
&\quad = (x_i - x_i') \int_X \Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i}(x)) \left.\frac{\partial \ln g(x|x_i)}{\partial x_i}\right|_{x_i = \hat{x}_i} g(x|\hat{x}_i) dx \\
&\quad = (x_i - x_i') \mathrm{Cov}\left(\Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i}(x)), \left.\frac{\partial \ln g(x|x_i)}{\partial x_i}\right|_{x_i = \hat{x}_i}\right)
\end{aligned} \tag{11}$$

where the last equality follows from the fact that

$$\mathbb{E}\left[\frac{\partial \ln g(x|x_i)}{\partial x_i}\right] = \int_X \frac{\partial \ln g(x|x_i)}{\partial x_i} g(x|x_i)dx = \int_X \frac{\partial g(x|x_i)}{\partial x_i} dx = 0$$

since $\int_X g(x|x_i)dx = 1$ is independent of x_i.

An upper bound on the covariance can be found as follows:

$$\left|\mathrm{Cov}\left(\Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i}(x)), \left.\frac{\partial \ln g(x|x_i)}{\partial x_i}\right|_{x_i=\hat{x}_i}\right)\right|$$

$$\leq \sqrt{\mathrm{Var}\left(\Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i}(x))\right)} \sqrt{\mathrm{Var}\left(\left.\frac{\partial \ln g(x|x_i)}{\partial x_i}\right|_{x_i=\hat{x}_i}\right)} \leq \kappa\iota \quad (12)$$

where the last inequality follows from the observation that assumption U3 together with $|a_i - a_i'| \leq 1$ and $|\boldsymbol{\alpha}_{-i} - \boldsymbol{\alpha}_{-i}'| \leq 1$ imply that $\mathrm{Var}\left(\Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i}(x))\right)$ is bounded by κ^2, and assumption 2 implies that $\mathrm{Var}[(\partial \ln g(x|x_i))/(\partial x_i)|_{x_i=\hat{x}_i}] \leq \iota$.

Equations (10), (11) and (12) together yield

$$\mathbb{E}[\Delta u_i(a_i, a_i', x_i, t_i, \boldsymbol{\mu}_{-i})] - \mathbb{E}[\Delta u_i(a_i, a_i', x_i', t_i, \boldsymbol{\mu}_{-i})] \geq (\delta - \kappa\iota)|x_i - x_i'|$$

which proves the lemma.

C Proof of Proposition 7

The following definition will be useful during the proof.

Definition 1 *Let $\mathcal{D}$ denote the set of distribution functions defined over $X \times A$ (i.e., the set of functions that are non-decreasing, left-continuous and have a range of $[0,1]$).*

Write the action set of player i as $A_i = \{a_0 = 0, a_1^i, a_2^i, \ldots, a_N^i = 1\}$ where $a_n^i < a_{n+1}^i$. For any given vector of distributions $\boldsymbol{\mu}_{-i}$, define $x_i(a_n; \boldsymbol{\mu}_{-i})$ by

$$\mathbb{E}[\Delta u_i(a_n, a_{n-1}, x_i(a_n; \boldsymbol{\mu}_{-i}), t_i, \boldsymbol{\mu}_{-i})] \triangleq 0 \quad (13)$$

i.e., the signal at which player i is indifferent between actions a_n and a_{n-1} for $n \in [1, N]$. Since the function Δu_i is strictly increasing in x_i if $\delta/\kappa > \iota$, $x_i(a_n; \boldsymbol{\mu}_{-i})$ is uniquely defined by equation (13). Player i's best response to the distribution $\boldsymbol{\mu}_{-i}$ can be represented as a step function with thresholds given by the finite vector $(x_i(a_n; \boldsymbol{\mu}_{-i}))_{n=1,\ldots,N}$; see Athey (2001).

Define

$$\chi_i(a, x; \boldsymbol{\mu}_{-i}) = \begin{cases} 1 & x < x_i(a; \boldsymbol{\mu}_{-i}) \\ 0 & x \geq x_i(a; \boldsymbol{\mu}_{-i}). \end{cases}$$

Then let

$$\phi_t(a, x; \boldsymbol{\mu}_{-t}) \equiv \int_{\{i|t_i=t\}} \chi_i(a, x; \boldsymbol{\mu}_{-i}) di$$

where $\boldsymbol{\mu}_{-t}$ is the vector of distributions induced by the strategies of all players who are not of type t, for $x \in X$ and $a \in A_t \equiv \cup_{\{i|t_i=t\}} A_i$. Notice that since ϕ_t is the sum of indicator functions, it is non-decreasing, left-continuous in (a, x) and maps $A_t \times X \to [0, 1]$. Hence ϕ_t is a distribution function i.e., $\phi_t \in \mathcal{D}$.

Let $\boldsymbol{\phi}(a, x; \boldsymbol{\mu}) \equiv (\phi_t)_{t \in T}$. $\boldsymbol{\phi}$ maps the set $\mathcal{D}^\tau$ into itself where τ is the number of types i.e., $\tau = \#T$. An equilibrium is defined by $\boldsymbol{\mu}(a, x) = \boldsymbol{\phi}(a, x; \boldsymbol{\mu})$.

Consider any two vectors of distribution functions $\boldsymbol{\mu} = (\mu_t)_{t \in T}$ and $\boldsymbol{\mu}' = (\mu_t')_{t \in T}$ defined over $A \times X$ where $A \equiv \cup_i A_i$. Let $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|$ denote the metric

$$\|\boldsymbol{\mu} - \boldsymbol{\mu}'\| \equiv \max_{t \in T} \max_{a \in A_t} \int_X |\mu_t(a, x) - \mu_t'(a, x)| dx. \tag{14}$$

(This is a variant of the L^1 metric, and so it is easy to show that it is indeed a metric.) The space $(\mathcal{D}, \|\|)$ is complete: by Helly's selection theorem (see Kolmogorov and Fomin (1970), p. 373), a sequence of non-decreasing, uniformly bounded functions on $X \subseteq \mathbb{R}$ has a subsequence that converges to a non-decreasing function. The objective is to establish that the mapping $\boldsymbol{\phi}$ is a contraction under the metric defined in equation (14). Existence and uniqueness of equilibrium then follow directly from the contraction mapping theorem.

Consider

$$\begin{aligned}
\|\phi_t(a,x;\boldsymbol{\mu}_{-t}) - \phi_t(a,x;\boldsymbol{\mu}'_{-t})\| &= \max_{a\in A_t} \int_X \left| \int_{\{i|t_i=t\}} \left(\chi_i(a,x;\boldsymbol{\mu}_{-i}) - \chi_i(a,x;\boldsymbol{\mu}'_{-i})\right) di \right| dx \\
&\leq \max_{a\in A_t} \int_X \int_{\{i|t_i=t\}} \left|\chi_i(a,x;\boldsymbol{\mu}_{-i}) - \chi_i(a,x;\boldsymbol{\mu}'_{-i})\right| di\ dx \\
&\leq \max_{a\in A_t} \int_{\{i|t_i=t\}} \int_X \left|\chi_i(a,x;\boldsymbol{\mu}_{-i}) - \chi_i(a,x;\boldsymbol{\mu}'_{-i})\right| dx\ di \\
&= \max_{a\in A_t} \int_{\{i|t_i=t\}} \left|x_i(a;\boldsymbol{\mu}_{-i}) - x_i(x,a;\boldsymbol{\mu}'_{-i})\right| di \\
&\leq \max_{a\in A_t} \sup_{\{i|t_i=t\}} \left|x_i(a;\boldsymbol{\mu}_{-i}) - x_i(x,a;\boldsymbol{\mu}'_{-i})\right|
\end{aligned}$$

for all $t \in T$. Since

$$\|\phi(a,x;\boldsymbol{\mu}) - \phi(a,x;\boldsymbol{\mu}')\| = \max_{t\in T} \|\phi_t(a,x;\boldsymbol{\mu}_{-t}) - \phi_t(a,x;\boldsymbol{\mu}'_{-t})\|,$$

a sufficient condition for $\boldsymbol{\phi}$ to be a contraction under the metric defined in equation (14) is therefore that

$$\max_{t\in T} \max_{a\in A_t} \sup_{\{i|t_i=t\}} \left|x_i(a;\boldsymbol{\mu}_{-i}) - x_i(x,a;\boldsymbol{\mu}'_{-i})\right| \leq \lambda\|\boldsymbol{\mu} - \boldsymbol{\mu}'\| \tag{15}$$

where $\lambda < 1$. Notice that the existence of dominance regions means that the distances defined by the metric exist and are finite.

Consider any two actions $a_n > a_{n-1}$ in the action set of player i. Then from equation (13),

$$\begin{aligned}
&\int_X \left(\Delta u_i(a_n,a_{n-1},x_i(a_n;\boldsymbol{\mu}'_{-i}),t_i,\boldsymbol{\mu}'_{-i}) - \Delta u_i(a_n,a_{n-1},x_i(a_n;\boldsymbol{\mu}_{-i}),t_i,\boldsymbol{\mu}'_{-i})\right) g(x|x_i(a_n;\boldsymbol{\mu}'_{-i}))dx \\
&\quad + \int_X \Delta u_i(a_n,a_{n-1},x_i(a_n;\boldsymbol{\mu}_{-i}),t_i,\boldsymbol{\mu}'_{-i}) \left(g(x|x_i(a_n;\boldsymbol{\mu}'_{-i})) - g(x|x_i(a_n;\boldsymbol{\mu}_{-i}))\right) dx \\
&+ \int_X \left(\Delta u_i(a_n,a_{n-1},x_i(a_n;\boldsymbol{\mu}_{-i}),t_i,\boldsymbol{\mu}'_{-i}) - \Delta u_i(a_n,a_{n-1},x_i(a_n;\boldsymbol{\mu}_{-i}),t_i,\boldsymbol{\mu}_{-i})\right) g(x|x_i(a_n;\boldsymbol{\mu}_{-i}))dx = 0;
\end{aligned}$$

or $\eta_1 + \eta_2 + \eta_3 = 0$. Hence $|\eta_1| = |\eta_2 + \eta_3| \leq |\eta_2| + |\eta_3|$. From assumptions U2–U4,

$$
\begin{aligned}
|\eta_1| &\geq \delta(a_n - a_{n-1})|x_i(a_n; \boldsymbol{\mu}_{-i})) - x_i(a_n; \boldsymbol{\mu}'_{-i}))|; \\
|\eta_2| &\leq \iota\kappa(a_n - a_{n-1})|x_i(a_n; \boldsymbol{\mu}_{-i})) - x_i(a_n; \boldsymbol{\mu}'_{-i}))|; \\
|\eta_3| &\leq \kappa\eta(a_n - a_{n-1}) \int_X |\boldsymbol{\mu}_{-i} - \boldsymbol{\mu}'_{-i}| dx
\end{aligned}
$$

where for any $x \in X$, $|\boldsymbol{\mu}_{-i} - \boldsymbol{\mu}'_{-i}| = \sup_{t \in T_{-i}} \sup_{a \in A_t} |\mu_t(x,a) - \mu'_t(x,a)|$.

Hence

$$
(\delta - \iota\kappa)|x_i(a_n; \boldsymbol{\mu}_{-i}) - x_i(a_n; \boldsymbol{\mu}'_{-i})| \leq \kappa\eta \int_X |\boldsymbol{\mu}_{-i} - \boldsymbol{\mu}'_{-i}| dx
$$

for all $a_n \in A_i$ and $i \in [0,1]$. Therefore

$$
\begin{aligned}
\max_{t \in T} \max_{a \in A_t} \sup_{\{i | t_i = t\}} \left| x_i(a; \boldsymbol{\mu}_{-i}) - x_i(a; \boldsymbol{\mu}'_{-i}) \right| &\leq \left(\frac{\kappa\eta}{\delta - \iota\kappa} \right) \max_{t \in T} \max_{a \in A_t} \sup_{\{i | t_i = t\}} \int_X |\boldsymbol{\mu}_{-i} - \boldsymbol{\mu}'_{-i}| dx \\
&\leq \left(\frac{\kappa\eta}{\delta - \iota\kappa} \right) \|\boldsymbol{\mu} - \boldsymbol{\mu}'\|.
\end{aligned}
$$

Hence the sufficient condition (15) for the contraction is satisfied if there is a number $\lambda > 1$ such that

$$
\frac{\kappa\eta}{\delta - \iota\kappa} \leq \lambda.
$$

The proposition follows.

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